

NONARCHIMEDEAN GEOMETRY, TROPICALIZATION, AND METRICS ON CURVES

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ABSTRACT. We develop a number of general techniques for comparing analytifications and tropicalizations of algebraic varieties. Our basic results include a projection formula for tropical multiplicities and a generalization of the Sturmfels-Tevelev multiplicity formula in tropical elimination theory to the case of a nontrivial valuation. For curves, we explore in detail the relationship between skeletal metrics and lattice lengths on tropicalizations and show that the maps from the analytification of a curve to the tropicalizations of its toric embeddings stabilize to an isometry on finite subgraphs. Other applications include generalizations of Speyer’s well-spacedness condition and the Katz-Markwig-Markwig results on tropical j -invariants.

1. INTRODUCTION

The recent work of Gubler [Gub07a, Gub07b], in addition to earlier work of Bieri-Groves [BG84], Berkovich [Ber90, Ber99, Ber04], and others, has revealed close connections between nonarchimedean analytic spaces (in Berkovich’s sense) and tropical geometry. One such connection is given by the second author’s theorem that ‘analytification is the inverse limit of all tropicalizations’ (see Theorem 1.2 below). This result is purely topological, providing a natural homeomorphism between the nonarchimedean analytification X^{an} of a quasiprojective variety X and the inverse limit of all ‘extended tropicalizations’ of X coming from closed immersions of X into quasiprojective toric varieties that meet the dense torus. In this paper, we develop a number of general techniques for comparing finer properties of analytifications and tropicalizations of algebraic varieties and apply these techniques to explore in detail the relationship between the natural metrics on analytifications and tropicalizations of curves. The proofs of our main results rely on the geometry of formal models and initial degenerations as well as Berkovich’s theory of nonarchimedean analytic spaces.

Let K be an algebraically closed field that is complete with respect to a nontrivial nonarchimedean valuation $\text{val} : K \rightarrow \mathbf{R} \cup \{\infty\}$. Let X be a nonsingular curve defined over K . The underlying topological space of X^{an} can be endowed with a ‘polyhedral’ structure locally modeled on an \mathbf{R} -tree. The leaves of X^{an} (i.e., the points at which there is a unique tangent direction in the sense of (5.65) below) are the K -points, together with the ‘type-4 points’ in Berkovich’s classification (3.10). The non-leaves are exactly those points that are contained in an embedded open segment, and the space $\mathbf{H}_o(X^{\text{an}})$ of non-leaves carries a canonical metric which, like the polyhedral structure, is defined using semistable models for X . See [Ber90, §4], [Thu05], [Bak08, §5], and §5 below, for details.

Suppose X is embedded in a toric variety Y_Δ and meets the dense torus \mathbf{T} . The tropicalization $\text{Trop}(X \cap \mathbf{T})$ is a 1-dimensional polyhedral complex with no leaves in the real vector space spanned by the lattice of one parameter subgroups of \mathbf{T} . All edges of $\text{Trop}(X \cap \mathbf{T})$ have slopes that are rational with respect to the lattice of one parameter subgroups, so there is a natural metric on $\text{Trop}(X \cap \mathbf{T})$ given locally by lattice length on each edge, and globally by shortest paths. The metric space $\mathbf{H}_o(X^{\text{an}})$ naturally surjects onto $\text{Trop}(X \cap \mathbf{T})$, but this map is far from being an isometry since infinitely many embedded segments in $\mathbf{H}_o(X^{\text{an}})$ are contracted. Furthermore, even when an edge of $\mathbf{H}_o(X^{\text{an}})$ maps homeomorphically onto an edge of $\text{Trop}(X \cap \mathbf{T})$, this homeomorphism need not be an isometry; see (2.5) below. Nevertheless, each embedded subgraph in $\mathbf{H}_o(X^{\text{an}})$ maps isometrically onto its image in all ‘sufficiently large’ tropicalizations.

Theorem 1.1. *Let Γ be a finite embedded subgraph in $\mathbf{H}_o(X^{\text{an}})$. Then there is a closed embedding of X into a quasiprojective toric variety such that X meets the dense torus and Γ maps isometrically onto*

its image in $\text{Trop}(X \cap \mathbf{T})$. Furthermore, the set of all such embeddings is stable and hence cofinal in the system of all embeddings of X into quasiprojective toric varieties whose images meet \mathbf{T} .

Here if $\iota : X \hookrightarrow Y_\Delta$ and $\iota' : X \hookrightarrow Y_{\Delta'}$ are closed embeddings into quasiprojective toric varieties such that X meets the dense tori \mathbf{T} and \mathbf{T}' , then we say that ι' *dominates* ι and we write $\iota' \geq \iota$ if there exists an equivariant morphism of toric varieties $\psi : Y_{\Delta'} \rightarrow Y_\Delta$ such that $\psi \circ \iota' = \iota$ (see (6.15.1)). In this case we have an induced map $\text{Trop}(X \cap \mathbf{T}') \rightarrow \text{Trop}(X \cap \mathbf{T})$; the above theorem says in particular that if Γ maps isometrically onto its image in $\text{Trop}(X \cap \mathbf{T})$, then the same is true for $\Gamma \rightarrow \text{Trop}(X \cap \mathbf{T}')$. In other words, the maps from $\mathbf{H}_o(X^{\text{an}})$ to the tropicalizations of toric embeddings of X stabilize to an isometry on every finite subgraph.

Both the analytification and the tropicalization constructions described above for subvarieties of tori globalize in natural ways. The analytification functor extends to arbitrary finite type K -schemes (see [Ber90, Chapters 2 and 3] or [Ber93]), and tropicalization extends to closed subvarieties of toric varieties as follows. If Δ is a fan in $N_{\mathbf{R}}$ and Y_Δ is the associated toric variety, then there is a natural ‘partial compactification’ $N_{\mathbf{R}}(\Delta)$ of $N_{\mathbf{R}}$ which is, set-theoretically, the disjoint union of the tropicalizations of all torus orbits in Y_Δ . The topology on $N_{\mathbf{R}}(\Delta)$ is such that the natural map from $Y_\Delta(K)$ extends to a continuous, proper, and surjective map $\text{trop} : Y_\Delta^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta)$. As in the case where Y_Δ is the torus \mathbf{T} , the tropicalization $\text{Trop}(X)$ of a closed subvariety X in Y_Δ is the closure of $\text{trop}(X(K))$ in $N_{\mathbf{R}}(\Delta)$, and the extended tropicalization map extends to a continuous, proper, surjective map from X^{an} onto $\text{Trop}(X)$. See [Pay09a, Rab12] and (4.2) below for further details.

Theorem 1.2. (Payne) *Let X be an irreducible quasiprojective variety over K . Then the inverse limit of the extended tropicalizations $\text{Trop}(\iota(X))$ over all closed immersions $\iota : X \hookrightarrow Y_\Delta$ into quasiprojective toric varieties is canonically homeomorphic to the analytification X^{an} .*

The inverse limit in Theorem 1.2 can be restricted to those closed immersions ι whose images meet the dense torus T_ι , and then the homeomorphism maps $X^{\text{an}} \setminus X(K)$ homeomorphically onto the inverse limit of the ordinary tropicalizations $\text{Trop}(\iota(X) \cap T_\iota)$.

When X is a curve, our Theorem 1.1 says that the metric structures on $\text{trop}(\iota(X) \cap T_\iota)$ stabilize to a metric on the subset $\mathbf{H}_o(X^{\text{an}})$ of the inverse limit, and the restriction of this homeomorphism is an *isometry*. In general, each sufficiently small segment e in $\mathbf{H}_o(X^{\text{an}})$ is mapped via an affine linear transformation with integer slope onto a (possibly degenerate) segment e' in $\text{Trop}(X)$. We write $m_{\text{rel}}(e)$ for the absolute value of the slope of this map, so if e has length ℓ then its image e' has lattice length $m_{\text{rel}}(e) \cdot \ell$. In Corollary 6.9, we relate these ‘expansion factors’ to *tropical multiplicities* of edges in $\text{Trop}(X)$. The notation is meant to suggest that $m_{\text{rel}}(e)$ may be thought of in this context as the relative multiplicity of e over e' . By definition, the tropical multiplicity $m_{\text{Trop}}(e')$ of an edge e' in a suitable polyhedral structure on $\text{Trop}(X)$ is the number of irreducible components, counted with multiplicities, in the *initial degeneration* $\text{in}_w(X \cap \mathbf{T})$ for any w in the relative interior of e' . These tropical multiplicities are fundamental invariants in tropical geometry and play a key role in the *balancing formula*. See (2.1) for a definition of the initial degeneration $\text{in}_w(X \cap \mathbf{T})$ and further discussion of tropical multiplicities, and Theorem 6.14 for a simple analytic proof of the balancing formula for smooth curves.

Theorem 1.3. *There is a polyhedral structure on $\text{Trop}(X \cap \mathbf{T})$ with the following properties.*

- (1) *For each edge e' in $\text{Trop}(X \cap \mathbf{T})$, there are finitely many embedded segments e_1, \dots, e_r in $\mathbf{H}_o(X^{\text{an}})$ mapping homeomorphically onto e' .*
- (2) *Any embedded segment in the preimage of e' that is disjoint from $e_1 \cup \dots \cup e_r$ is contracted to a point.*
- (3) *The tropical multiplicity of e' is the sum of the corresponding expansion factors*

$$m_{\text{Trop}}(e') = m_{\text{rel}}(e_1) + \dots + m_{\text{rel}}(e_r).$$

The properties above are preserved by subdivision, so they hold for any sufficiently fine polyhedral structure on $\text{Trop}(X \cap \mathbf{T})$. See Proposition 6.4 and Corollary 6.9.

The tropical multiplicity formula in the above theorem gives an important connection to nonarchimedean analytic spaces that is not visible from the definitions. The formula shows, for example, that if e' is a small segment in $\text{Trop}(X \cap \mathbf{T})$ whose tropical multiplicity is equal to 1, then there is a unique segment e in $\mathbf{H}_o(X^{\text{an}})$ mapping homeomorphically onto e' , and the length of e is equal to the tropical length of e' . It is well known that the skeleton of the analytification of an elliptic curve with bad reduction is a loop of length equal to minus the valuation of the j -invariant (see Remark 5.51), so these formulas explain earlier results of Katz, Markwig and Markwig on tropical j -invariants of genus one curves in toric surfaces [KMM08, KMM09]. See, for instance, Example 2.10. The following theorem also provides natural generalizations for genus one curves in higher dimensional toric varieties, as well as curves of arbitrary genus.

Theorem 1.4. *Let Γ' be a finite embedded subgraph of $\text{Trop}(X \cap \mathbf{T})$ and suppose $\text{in}_w(X \cap \mathbf{T})$ is irreducible and generically reduced for every w in Γ' . Then there is a unique embedded subgraph Γ in $\mathbf{H}_o(X^{\text{an}})$ mapping homeomorphically onto Γ' , and this homeomorphism is an isometry.*

See §7 for details on deducing the tropical j -invariant results of Katz, Markwig and Markwig from the above theorem, applications to certifying faithful tropical representations of minimal skeleta, and examples illustrating these applications.

The expansion factors $m_{\text{rel}}(e)$ in our tropical multiplicity formula are often computable in practice. If X is an affine curve embedded in the torus \mathbf{G}_m^n via an n -tuple of invertible regular functions f_1, \dots, f_n , then

$$m_{\text{rel}}(e) = \gcd(s_1(e), \dots, s_n(e)),$$

where $s_i(e)$ is the absolute value of the slope of the integer-affine function $\log |f_i|$ along the edge e . See Remark 6.6. The quantities $s_i(e)$ are easily calculated from the divisors of f_1, \dots, f_n using the ‘Slope Formula’ given in Theorem 5.69.

In concrete situations, it is useful to be able to certify that a given tropicalization map faithfully represents a large piece of the nonarchimedean analytification X^{an} (e.g. the ‘minimal skeleton’ Σ of X^{an} in the sense of Berkovich [Ber90] or Corollary 5.50) using only ‘tropical’ computations (e.g. Gröbner complex computations which have been implemented in computer algebra packages such as Singular or Macaulay2), as opposed to calculations with formal models that have not been implemented in a systematic way in any existing software package. We prove that a tropicalization map represents Σ faithfully, meaning that the map is an isometry on Σ , provided that certain combinatorial and topological conditions are satisfied. Our results on faithful representations are presented in conjunction with some observations about initial degenerations which help explain the special role played by trivalent graphs in the literature on tropical curves (cf. Theorem 6.25 and Remark 6.27).

We explore tropicalizations of elliptic curves in detail as a concrete illustration of our methods and results. We are able to say some rather precise things in this case; for example, we show that every elliptic curve E/K with multiplicative reduction admits a closed embedding in \mathbf{P}^2 whose tropicalization faithfully (and certifiably) represents the minimal skeleton of E^{an} . We show that E also has tropicalizations in which parts of the minimal skeleton are expanded, parts are contracted, or both. As mentioned above, we are also able to recover and generalize many of the results of Katz, Markwig and Markwig on tropical j -invariants. Furthermore, we interpret Speyer’s ‘well-spacedness condition’ for trivalent tropicalizations of totally degenerate genus 1 curves [Spe07] as a statement about rational functions on the analytification of the curve, and prove generalizations of this condition for nontrivalent tropicalizations, and for genus 1 curves with good reduction.

The paper concludes with a discussion of tropical elimination theory and some new methods for computing tropicalizations of curves which are afforded by our results. One reason to be interested in computing tropicalizations is that it is useful for the method of ‘tropical implicitization’ [ST08, STY07], which attempts to shed light on the practical problem of finding implicit equations for parametrically represented varieties (see (8.8) for further discussion). From the theoretical point of view, at least, implicitization is a special case of elimination theory, and the tropical approach to elimination theory

pioneered by Sturmfels and Tevelev is of both practical and theoretical interest. The basic result in tropical elimination theory is the Sturmfels-Tevelev multiplicity formula, which calculates $\text{Trop}(\alpha(X))$ (as a weighted polyhedral complex) in terms of $\text{Trop}(X)$ when $\alpha : \mathbf{T} \rightarrow \mathbf{T}'$ is a homomorphism of tori which induces a generically finite map from a subvariety X in \mathbf{T} onto its image. The multiplicity formula in [ST08] is formulated and proved in the ‘constant coefficient’ setting, where K is the field of Puiseux series over an algebraically closed coefficient field k of characteristic 0 and X is defined over k . We generalize the Sturmfels-Tevelev formula to the case where X is any closed subvariety of a torus \mathbf{T} defined over a complete and algebraically closed nonarchimedean field K .

Philosophically speaking, there are at least two long-term goals to this paper. On the one hand, we believe that the systematic use of modern tools from nonarchimedean geometry is extremely useful for understanding and proving theorems in tropical geometry. This paper takes several steps in that direction, establishing some new results in tropical geometry via Berkovich’s theory and the Bosch-Lütkebohmert-Raynaud theory of admissible formal schemes. On the other hand, much of this paper can be viewed as a comparison between two different ways of approximating nonarchimedean analytic spaces. Nonarchimedean analytic spaces have proved to be useful in many different contexts, but the topological spaces underlying them are wildly branching infinite complexes which are difficult to study directly, so one usually approximates them with finite polyhedral complexes. One such approximation goes through skeleta of nice (e.g. semistable) formal models (cf. Theorem 5.57 below), another through (extended) tropicalizations (cf. Theorem 1.2). Our Theorem 6.21 shows that, in the case of curves, these two approximations have the same metric structure in the limit (though the metrics may be different at any given finite level).

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2. BASIC NOTIONS AND EXAMPLES

Here we give a brief overview of the basic notions necessary to understand the theorems stated in the introduction, followed by a few key examples illustrating these results. Throughout this paper, K is an algebraically closed field that is complete with respect to a nontrivial nonarchimedean valuation

$$\text{val} : K \rightarrow \mathbf{R} \cup \{\infty\}.$$

We let $G = \text{val}(K^\times)$ be its value group, $R = \text{val}^{-1}([0, \infty])$ its valuation ring, $\mathfrak{m} \subset R$ the maximal ideal, and $k = R/\mathfrak{m}$ its residue field (which is algebraically closed by [Rob00, §2.1 Proposition 3]). Let $|\cdot| = \exp(-\text{val}(\cdot))$ be the absolute value on K associated to the valuation.

2.1. Tropicalization. Let M be a free abelian group of rank n , let $\mathbf{T} = \text{Spec}(K[M])$ be the K -torus with character group M , and let $N = \text{Hom}(M, \mathbf{Z})$ be the dual lattice. If X is a closed subscheme of \mathbf{T} , there is a natural *tropicalization map*

$$\text{trop} : X(K) \rightarrow N_{\mathbf{R}},$$

where $N_{\mathbf{R}} = \text{Hom}(M, \mathbf{R})$. The image of a point x in $X(K)$ is the linear function taking $u \in M$ to the valuation of the corresponding character evaluated at x . Then $\text{Trop}(X)$ is the closure of $\text{trop}(X(K))$ in the Euclidean topology on $N_{\mathbf{R}}$. Note that the choice of an isomorphism $M \cong \mathbf{Z}^n$ induces an identification of \mathbf{T} with \mathbf{G}_m^n . In such coordinates, the tropicalization map sends a point (x_1, \dots, x_n) in $X(K)$ to $(\text{val}(x_1), \dots, \text{val}(x_n))$ in \mathbf{R}^n .

One of the basic results in tropical geometry says that if X is an integral subscheme of \mathbf{T} of dimension d then $\text{Trop}(X)$ is the underlying set of a connected ‘balanced weighted integral G -affine polyhedral complex’ of pure dimension d . We do not define all of these terms here, but briefly recall how one gets a polyhedral complex and defines weights on the maximal faces of this complex. Let w be a point in $N_G = \text{Hom}(M, G)$. The ‘tilted group ring’ $R[M]^w$ is the subring of $K[M]$ consisting of Laurent polynomials $a_1 x^{u_1} + \dots + a_r x^{u_r}$ such that

$$\text{val}(a_i) + \langle u_i, w \rangle \geq 0$$

for all i . The R -scheme $\mathbf{T}^w = \text{Spec } R[M]^w$ is a torsor for the torus $\text{Spec } R[M]$, and its generic fiber is canonically isomorphic to \mathbf{T} . If X is a closed subscheme of \mathbf{T} defined by an ideal $\mathfrak{a} \subset K[M]$ then

$$X^w = \text{Spec } (R[M]^w / (\mathfrak{a} \cap R[M]^w))$$

is a flat R -scheme with generic fiber X , which we call the *tropical integral model* associated to w . It is exactly the closure of X in \mathbf{T}^w . The special fiber $\text{in}_w(X)$ of X^w is called the *initial degeneration* of X with respect to w and is the subscheme of the special fiber of \mathbf{T}^w cut out by the w -initial forms of Laurent polynomials in \mathfrak{a} , in the sense of generalized Gröbner theory.

The scheme \mathbf{T}^w is not proper, so points in $X(K)$ may fail to have limits in the special fiber. Indeed, the special fiber $\text{in}_w(X)$ is often empty. One of the fundamental theorems in tropical geometry says that w is in $\text{Trop}(X)$ if and only if $\text{in}_w(X)$ is not empty.¹ Moreover, $\text{Trop}(X)$ can be given the structure of a finite polyhedral complex in such a way that whenever w and w' belong to the relative interior of the same face, the corresponding initial degenerations $\text{in}_w(X)$ and $\text{in}_{w'}(X)$ are \mathbf{T} -affinely equivalent.

We define the *multiplicity* $m_{\text{Trop}}(w)$ of a point w in $\text{Trop}(X)$ to be the number of irreducible components of $\text{in}_w(X)$, counted with multiplicities. In particular $m_{\text{Trop}}(w) = 1$ if and only if $\text{in}_w(X)$ is

¹In the special case where \mathbf{T} has dimension one and X is the zero locus of a Laurent polynomial f , this is equivalent to the statement that f has a root with valuation s if and only if $-s$ is a slope of the Newton polygon of f .

irreducible and generically reduced. These tropical multiplicities are constant on the relative interior of each face F of $\text{Trop}(X)$, and we define the multiplicity $m_{\text{Trop}}(F)$ to be $m_{\text{Trop}}(w)$ for any w in the relative interior of F . The multiplicities for maximal faces are the ‘weights’ mentioned above that appear in the balancing condition. These weights have the following simple interpretation for hypersurfaces.

Remark 2.2. If $X = V(f)$ is a hypersurface then $\text{Trop}(X)$ is the corner locus of the convex piecewise linear function associated to a defining equation f [EKL06, Section 2.1]. In this case, $\text{Trop}(X)$ has a unique minimal polyhedral structure, and the initial degenerations are essentially constant on the relative interior of each face. There is a natural inclusion reversing bijection between the faces of $\text{Trop}(X)$ in this minimal polyhedral structure and the positive dimensional faces of the *Newton polytopal complex* (or *Newton complex*) of f : a face of $\text{Trop}(X)$ corresponds to the convex hull of the monomials whose associated affine linear function is minimal on that face. In particular, the maximal faces of $\text{Trop}(X)$ correspond to the edges of this Newton complex. In this special case, the multiplicity of a maximal face is the lattice length of the corresponding edge.² The relationship between the tropical hypersurface and the Newton complex is also explained in more detail in [Rab12, §8].

2.3. Analytification. Let A be a finite-type K -algebra. The *Berkovich spectrum* of A , denoted $\mathcal{M}(A)$, is defined to be the set of multiplicative seminorms $\|\cdot\|$ on A extending the absolute value $|\cdot|$ on K . The definition of a multiplicative seminorm is given in (3.2) below. The Berkovich spectrum $\mathcal{M}(A)$ is the underlying set of the nonarchimedean analytification X^{an} of $X = \text{Spec}(A)$. The topology on X^{an} is the coarsest such that the map $\|\cdot\| \mapsto \|f\|$ is continuous for every $f \in A$; this coincides with the subspace topology induced by the inclusion of X^{an} in \mathbf{R}^A .

Remark. We will often write \mathbf{A}_{an}^1 for $\mathbf{A}^{1,\text{an}}$ and \mathbf{P}_{an}^1 for $\mathbf{P}^{1,\text{an}}$, etc.

If X is connected then X^{an} is a path-connected locally compact Hausdorff space that naturally contains $X(K)$ as a dense subset; a point $x \in X(K)$ corresponds to the seminorm $\|\cdot\|_x$ given by $\|f\|_x = |f(x)|$. The analytification procedure $X \mapsto X^{\text{an}}$ gives a covariant functor from the category of locally finite-type K -schemes to the category of topological spaces.³

If X is a closed subvariety of \mathbf{T} then the tropicalization map described in the previous section extends from $X(K)$ to a continuous and proper map

$$\text{trop} : X^{\text{an}} \rightarrow N_{\mathbf{R}}$$

taking a seminorm $\|\cdot\|$ to the linear function $u \mapsto -\log \|x^u\|$, and the image of this map is exactly $\text{Trop}(X)$. In other words, $\text{trop}(X^{\text{an}})$ is the closure of the image of $X(K)$ in $N_{\mathbf{R}}$.

2.4. Metric structure of analytic curves. Following [Bak08], we describe the metric structure on the analytification of the affine line, and then indicate how one generalizes this to arbitrary algebraic curves. This material is presented in much more detail in §5 below.

As mentioned in the previous section, a point x in $\mathbf{A}^1(K)$ determines a multiplicative seminorm $\|f\|_x = |f(x)|$ on $K[T]$, which is a point in the analytification \mathbf{A}_{an}^1 . More generally, closed balls of nonnegative radius determine multiplicative seminorms as follows. Fix $x \in \mathbf{A}^1(K)$ and a radius $r \geq 0$, and let $B = \mathbf{B}(x, r)$ be the closed ball

$$\mathbf{B}(x, r) = \{y \in \mathbf{A}^1(K) : |x - y| \leq r\}.$$

Then the supremum

$$\|f\|_B = \sup_{y \in B} |f(y)|$$

is a multiplicative seminorm on $K[T]$. Distinct balls correspond to distinct seminorms, i.e. distinct points of \mathbf{A}_{an}^1 . According to *Berkovich’s classification theorem* [Ber90, p. 18], every point in \mathbf{A}_{an}^1 is

²This is explained in [ST08, Example 3.16] in the special case where X is irreducible, K is the field of Puiseux series over k , and X is defined over k . The arguments given there work in full generality.

³The analytification X^{an} has additional structure, including a structure sheaf, and $X \mapsto X^{\text{an}}$ may also be seen as a functor from locally finite-type K -schemes to locally ringed spaces. See [Ber90, §2.3, §3.1, §3.4] or (3.2) for more details.

the limit of a sequence of points $\|\cdot\|_{B_n}$ corresponding to a nested sequence $B_1 \supset B_2 \supset \cdots$ of balls of positive radius. The analytification \mathbf{P}_{an}^1 of \mathbf{P}^1 over K , as a topological space, is just the one-point compactification of \mathbf{A}_{an}^1 , with the extra point corresponding naturally to ∞ in $\mathbf{P}^1(K)$.

The analytification of \mathbf{P}^1 is uniquely path connected, which means that there is a unique embedded path between any two distinct points. In the important special case where these points are $\|\cdot\|_B$ and $\|\cdot\|_{B'}$, where $B' = \mathbf{B}(x', r')$ is disjoint from B , this path can be described explicitly as follows. Let $R = |x - x'|$. Then $\mathbf{B}(x, R) = \mathbf{B}(x', R)$ is the smallest ball that contains both B and B' , and

$$\{\|\cdot\|_{\mathbf{B}(x, \rho)} : r \leq \rho \leq R\} \cup \{\|\cdot\|_{\mathbf{B}(x', \rho')} : R \geq \rho' \geq r'\}$$

is the image of the unique embedded path from $\|\cdot\|_B$ to $\|\cdot\|_{B'}$.

There is a natural metric on $\mathbf{P}_{\text{an}}^1 \setminus \mathbf{P}^1(K)$ in which, for $r < R$, the distance between $\|\cdot\|_{\mathbf{B}(x, r)}$ and $\|\cdot\|_{\mathbf{B}(x, R)}$ is $\log(R/r)$, which is the *modulus* of the analytic open annulus $\{y \in \mathbf{A}^1(K) : r < |x - y| < R\}$ (see §5). We write $\mathbf{H}(\mathbf{P}_{\text{an}}^1)$ to denote $\mathbf{P}_{\text{an}}^1 \setminus \mathbf{P}^1(K)$ with this metric structure.⁴

The metric on $\mathbf{H}(\mathbf{P}_{\text{an}}^1)$ has the important property that, roughly speaking, $\log|f|$ is piecewise affine with integer slopes for any nonzero rational function $f \in K(T)$. More precisely, suppose that f is nonconstant and that $\widehat{\Sigma}$ is the minimal closed connected subset of \mathbf{P}_{an}^1 containing the set S of zeros and poles of f . Let $\Sigma = \widehat{\Sigma} \setminus S$. Then:

- (1) The subspace Σ of $\mathbf{H}(\mathbf{P}_{\text{an}}^1)$ is a metric graph with finitely many edges, in which the edges whose closures meet K have infinite length.
- (2) The restriction of $\log|f|$ to Σ is piecewise affine with integer slopes.
- (3) There is a natural retraction map from \mathbf{P}_{an}^1 onto $\widehat{\Sigma}$.
- (4) The function $\log|f|$ from \mathbf{P}_{an}^1 to $\mathbf{R} \cup \{\pm\infty\}$ factors through the retraction onto $\widehat{\Sigma}$, and hence is determined by its restriction to Σ .

The metric on the complement of the set of K -points in the analytification of an arbitrary algebraic curve is induced by the metric on $\mathbf{P}_{\text{an}}^1 \setminus \mathbf{P}^1(K)$ via semistable decomposition. See Section 5 for details.

There is also a notion of a skeleton of a smooth and connected but not necessarily complete curve X . Let \widehat{X} be the smooth compactification of X and let $D = \widehat{X} \setminus X$ be the set of ‘punctures’. Choose a semistable model \mathcal{X} of \widehat{X} such that the punctures reduce to distinct smooth points of the special fiber $\overline{\mathcal{X}}$. Then there is unique minimal closed connected subset Σ of X^{an} containing the skeleton $\Sigma_{\mathcal{X}}$ of \widehat{X} and whose closure in \widehat{X}^{an} contains D . We call Σ the *skeleton* of X associated to \mathcal{X} . As above, there is a canonical retraction map $\tau_{\Sigma} : X^{\text{an}} \rightarrow \Sigma$. If $X \subset \mathbf{T}$ then the tropicalization map $\text{trop} : X \rightarrow N_{\mathbf{R}}$ factors through τ_{Σ} . There is a skeleton which is minimal over all models \mathcal{X} if $2 - 2g(\widehat{X}) - \#D \leq 0$. See §5 for a complete discussion of the skeleta of a curve.

2.5. Examples. To illustrate our main results concerning the relationship between analytification and tropicalization in the case of curves, taking into account the metric structure on both sides, we present the following examples. In each example, we fix a specific coefficient field for concreteness.

Our first example illustrates the necessity of subdividing $\text{Trop}(X)$ in Theorem 1.3. The theorem says that each edge of multiplicity two in $\text{Trop}(X)$ has a subdivision such that each edge is either the isometric image of two distinct edges in $\mathbf{H}_{\circ}(X^{\text{an}})$ or the 2-fold dilation of a single edge. We show that both phenomena can happen in a single edge of $\text{Trop}(X)$ on which the initial degenerations are constant, and hence such edges must be subdivided for the theorem to hold. This example is further explained in Examples 8.7 and 8.11.

Example 2.6. Let p be a prime, and let K be the field \mathbf{C}_p of ‘ p -adic complex numbers’, with the valuation normalized so that $\text{val}(p) = 1$. Consider the curve X in \mathbf{G}_m^2 defined parametrically by $x(t) = t(t - p)$ and $y(t) = t - 1$. The image is cut out by the equation $y^2 + (2 - p)y = x + (p - 1)$. The tropicalization $\text{Trop}(X)$ consists of 3 rays emanating from the origin in the directions $(1, 0)$, $(0, 1)$, and $(-2, -1)$. The ray in the direction $(1, 0)$ has tropical multiplicity 2, because the initial degeneration

⁴It is important to note that the metric topology on $\mathbf{H}(\mathbf{P}_{\text{an}}^1)$ is much finer than the subspace topology on $\mathbf{P}_{\text{an}}^1 \setminus \mathbf{P}^1(K)$. Our notation follows [BR10] and reflects the fact that the metric on $\mathbf{H}(\mathbf{P}_{\text{an}}^1)$ is 0-hyperbolic in the sense of Gromov.

at any point $(\alpha, 0)$ with $\alpha > 0$ is isomorphic to the non-reduced scheme $\text{Spec } \overline{\mathbf{F}}_p[x, x^{-1}, y]/(y+1)^2$, which is a length 2 nilpotent thickening of the torus $\text{Spec } \overline{\mathbf{F}}_p[x, x^{-1}]$. The other two rays have tropical multiplicity 1.

The tropicalization map $\text{trop} : X^{\text{an}} \rightarrow \text{Trop}(X)$ factors through the retraction map onto the minimal skeleton Σ of X , shown in Figure 1.

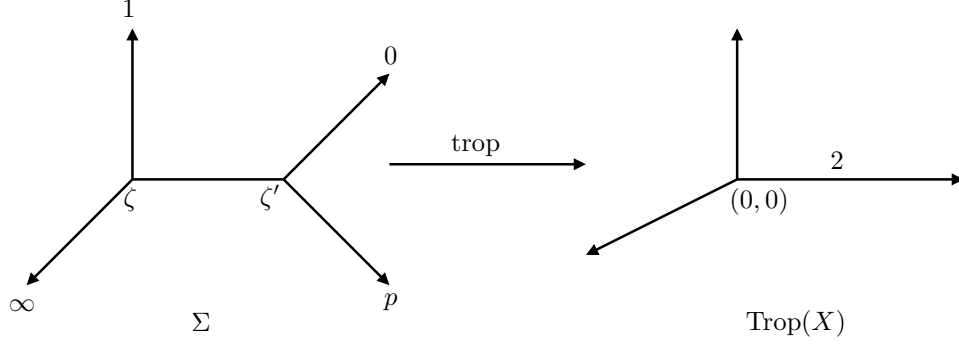


FIGURE 1. An illustration of the tropicalization of the curve X from Example 2.6. The ray $\{(\alpha, 0) : \alpha > 0\} \subset \text{Trop}(X)$ has tropical multiplicity 2.

The parametrization map identifies X with $\mathbf{P}^1 \setminus \{0, 1, p, \infty\}$, and the trivalent vertices of Σ are then

$$\zeta = \|\cdot\|_{\mathbf{B}(0,1)} \text{ and } \zeta' = \|\cdot\|_{\mathbf{B}(0, \frac{1}{p})}$$

in \mathbf{A}_{an}^1 . The edge e connecting them has length 1, but the image of e under trop is the segment in \mathbf{R}^2 connecting $(0, 0)$ to $(2, 0)$, which has lattice length 2. Thus the expansion factor $m_{\text{rel}}(e)$ is 2. On the other hand, for any closed segment e' contained in one of the four unbounded rays of Σ (leading from either ζ toward 1 or ∞ or from ζ' toward 0 or p), trop maps e' isometrically onto its image.

In this example, the point $(2, 0)$ in $\text{Trop}(X)$ is ‘special’ from the nonarchimedean analytic point of view — it is the image under trop of a trivalent vertex of Σ — but there is nothing special about $(2, 0)$ from the ‘classical’ tropical perspective, as the initial degenerations are constant along the entire positive x -axis. See Example 2.7 for another example of this phenomenon.

When X is a curve in \mathbf{G}_m^n , there are several natural polyhedral structures (i.e. decompositions into vertices and segments) for $\text{Trop}(X)$:

- (1) The minimal polyhedral structure on $\text{Trop}(X)$, in which all vertices have valence at least 3.
- (2) The “analytic” polyhedral structure, in which the vertices are the images of vertices of valence at least 3 in the minimal skeleton Σ of X^{an} .
- (3) The “tropical” polyhedral structure, in which the vertices are the points w such that $\text{in}_w(X)$ is not invariant under the action of a 1-parameter subgroup.

The preceding example shows that (2) may be strictly finer than (1) and (3). The following example, due to Dustin Cartwright, shows that (3) can be strictly finer than (1). Cartwright has also given an example showing that (3) can be strictly finer than (2). See [Car12].

Example 2.7. Let K be the completion of the field of Puiseux series $\mathbf{C}\{\{t\}\}$ and consider the curve X in \mathbf{G}_m^3 cut out by the 2×2 minors of the matrix

$$\begin{bmatrix} t(x-1) & t(y-1) & z-1 \\ t(y-1) & z-1 & 1 \end{bmatrix}.$$

Its defining ideal is generated by $x-1-(y-1)(z-1)$ and $(z-1)^2-t(y-1)$. It is a rational curve, given parametrically by $u \mapsto (u^3/t+1, u^2/t+1, u+1)$. From this parametric description, one sees that

X is isomorphic to \mathbf{P}^1 minus 7 points, specifically, the cube roots Q_1, Q_2, Q_3 of $-t$, the square roots P_1, P_2 of $-t$, and -1 and ∞ . Let $a, b, c \in \mathbf{P}_{\text{an}}^1$ be the supremum norms $\|\cdot\|_{\mathbf{B}(0,1)}$, $\|\cdot\|_{\mathbf{B}(0,|t|^{1/3})}$, and $\|\cdot\|_{\mathbf{B}(0,|t|^{1/2})}$, respectively. The minimal skeleton Σ of X^{an} then consists of the two bounded segments \overline{ab} and \overline{bc} , of lengths $1/3$ and $1/6$, respectively, with two rays emanating from a (pointing toward -1 and ∞), three rays emanating from b (pointing toward Q_1, Q_2, Q_3) and two rays emanating from c (pointing toward P_1, P_2). See Figure 2.

The tropicalization $\text{Trop}(X)$ consists of two vertices, one segment, and four rays. The vertices are $v_1 = (-1, -1, 0) = \text{trop}(a)$ and $v_2 = (0, -1/3, 0) = \text{trop}(b)$. The segment $\overline{v_1 v_2}$ has multiplicity 1. The rays based at v_1 are in directions $(0, 0, 1)$ and $(-3, -2, -1)$, both with multiplicity 1. The rays based at v_2 are in directions $(0, 1, 0)$ and $(1, 0, 0)$ with multiplicities 2 and 3, respectively.

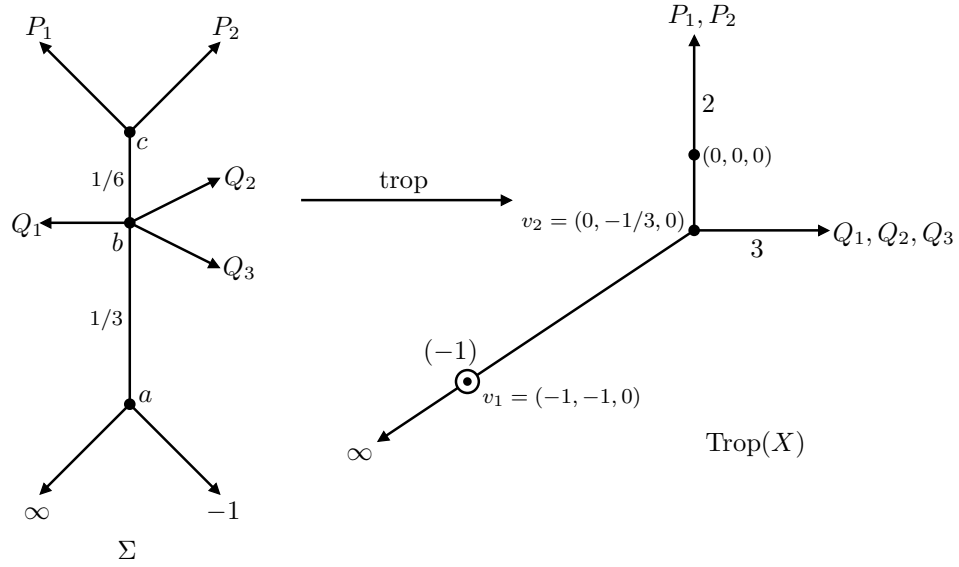


FIGURE 2. The skeleton and tropicalization of the rational curve X of Example 2.7. The points a, b, c of Σ correspond to the supremum norms $\|\cdot\|_{\mathbf{B}(0,1)}$, $\|\cdot\|_{\mathbf{B}(0,|t|^{1/3})}$, and $\|\cdot\|_{\mathbf{B}(0,|t|^{1/2})}$, respectively; their images in $\text{Trop}(X)$ are v_1 , v_2 , and $(0, 0, 0)$, respectively. The points P_1, P_2 are the square roots of t and Q_1, Q_2, Q_3 are the cube roots of t . The circled dot at v_1 is meant to indicate that there is a ray coming out of the page in the direction $(0, 0, 1)$; the tropicalization of the ray towards ∞ emanates from v_1 and goes into the page in the direction $(-3, -2, -1)$. The rest of $\text{Trop}(X)$ is contained in the (x, y) -plane.

From this description, one sees that the point $(0, 0, 0)$ is special from the analytic point of view, because it is equal to $\text{trop}(c)$. The initial degeneration at $(0, 0, 0)$ is defined by $x - 1 - (y - 1)(z - 1)$ and $(z - 1)^2$, which is a two fold thickening of the 1-parameter subgroup in the y -direction, but not invariant under the action of the subgroup. The point $(0, 0, 0)$ divides the vertical ray into two regions, the open segment from the origin to v_2 , along which the initial degeneration is generated by $z - 1$ and $(x - 1)^2$, and the open ray in direction $(0, 1, 0)$, along which the initial degeneration is generated by $x - z$ and $(z - 1)^2$. In particular the tropical polyhedral structure is strictly finer than the minimal polyhedral structure on $\text{Trop}(X)$.

Our next example shows how a loop in the analytification of a genus 1 curve can be collapsed onto a segment of multiplicity greater than 1.

Example 2.8. Let K be the completion of the field of Puiseux series $\mathbf{C}\{\{t\}\}$. Consider the genus 1 curve $\widehat{E} \subset \mathbf{P}^2$ over K defined by the Weierstrass equation $y^2 = x^3 + x^2 + t^4$, and let $E = \widehat{E} \cap \mathbf{G}_m^2$.

The j -invariant of \widehat{E} has valuation -4 , so \widehat{E} has multiplicative reduction and the minimal skeleton Σ of \widehat{E} is isometric to a circle of circumference 4. In this example, $\text{Trop}(E)$ does not have a cycle even though \widehat{E}^{an} does; it is interesting to examine exactly what tropicalization is doing to \widehat{E}^{an} . Let Γ be the minimal skeleton of E ; as above, trop factors through the retraction of E^{an} onto Γ . Figure 3 shows the restriction of trop to Γ .

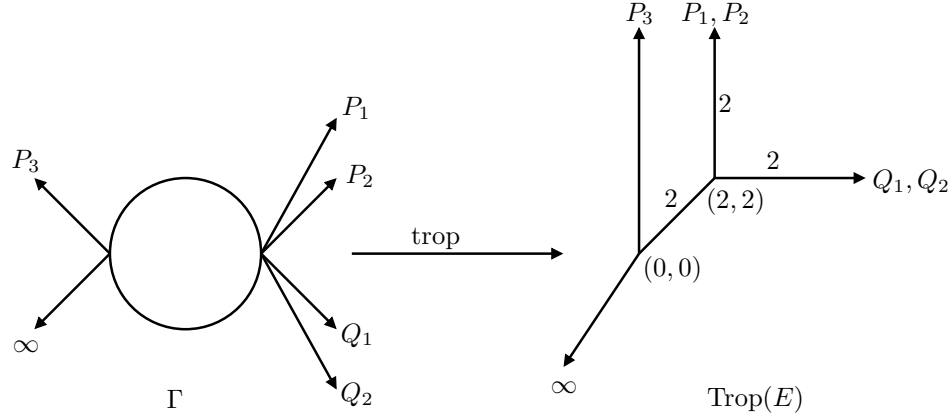


FIGURE 3. The minimal skeleton $\Gamma \subset E^{\text{an}}$ and the tropicalization $\text{Trop}(E)$ from Example 2.8. The points P_i, Q_j are defined as follows: the rational function x on \widehat{E} has divisor $(Q_1) + (Q_2) - 2(\infty)$ where $Q_1 = (0, t^2)$ and $Q_2 = (0, -t^2)$, and y has divisor $(P_1) + (P_2) + (P_3) - 3(\infty)$, where $\text{val}(x(P_1)) = \text{val}(x(P_2)) = 2$ and $\text{val}(x(P_3)) = 0$.

The tropicalization map sends Σ 2-to-1 onto its image in $\text{Trop}(E)$, which is a segment of tropical length 2 and tropical multiplicity 2. Locally on Σ the tropicalization map is an isometry. Each of the rays of Γ emanating from Σ maps isometrically onto its image. The two rays in $\text{Trop}(E)$ with multiplicity 1 have unique preimages in Γ , while there are two distinct rays in Γ mapping onto each of the two rays in $\text{Trop}(E)$ of multiplicity 2. See Example 8.13 for more details on these computations.

One can also consider tropicalizations of the genus 1 curve \widehat{E} in Example 2.8 associated to other embeddings of E into complete toric varieties (e.g. products of projective spaces). In §7 we will see that there are tropicalizations of E for which: (a) Σ maps homeomorphically and isometrically onto its image; (b) Σ is mapped to a single point; (c) Σ is mapped to a cycle of length 3 (a segment of Σ having length 1 gets collapsed to a point); (d) Σ is mapped to a cycle of length 5 (a segment of Σ having length 1 gets mapped to a segment of tropical length 2); and (e) Σ is mapped to a cycle of length 4 ‘by accident’ (one segment of Σ having length 1 gets collapsed to a point, another gets mapped to a segment of tropical length 2). Moreover, these constructions will be generalized to arbitrary genus 1 curves with multiplicative reduction.

The following example illustrates a different kind of collapse, where a segment e in the minimal skeleton of the analytification is collapsed to a point, i.e. the relative multiplicity $m_{\text{rel}}(e)$ is zero.

Example 2.9. Let $p \geq 5$ be a prime, let $K = \mathbf{C}_p$, and let $k \cong \overline{\mathbf{F}}_p$ be its residue field. Let $X \subset \mathbf{G}_m^2$ be the affine curve over K defined by the equation $f(x, y) = x^3y - x^2y^2 - 2xy^3 - 3x^2y + 2xy - p = 0$. The curve $\widehat{X} \subset \mathbf{P}^2$ defined by the homogenization $\widehat{f}(x, y, z) = x^3y - x^2y^2 - 2xy^3 - 3x^2yz + 2xyz^2 - pz^4 = 0$ is a smooth plane quartic of genus 3, and the given equation $\widehat{f}(x, y, z) = 0$ defines the minimal regular proper semistable model \mathcal{X} for \widehat{X} over \mathbf{Q}_p . The special fiber $\overline{\mathcal{X}}$ of \mathcal{X} consists of four (reduced) lines in general position in \mathbf{P}_k^2 , since $\widehat{f} \bmod p$ factors as $xy(x + y - z)(x - 2y - 2z)$. The tropicalization $\text{Trop}(X) \subset \mathbf{R}^2$ consists of a triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ together with three rays emanating from these three vertices in the directions of $(-1, -1)$, $(3, -1)$, $(-1, 3)$ respectively.

The three bounded edges/rays incident to $(0, 0)$ all have tropical multiplicity 2, and all other bounded edges/rays in $\text{Trop}(X)$ have tropical multiplicity 1. Let Σ be the skeleton Σ_X of \hat{X} and let Γ be the minimal skeleton of X . Then Σ is a tetrahedron (with vertices corresponding to the four irreducible components of \bar{X}) with six edges of length 1, because this is the dual graph of a regular semistable model defined over \mathbf{Z}_p , and Γ is obtained from Σ by adding a ray emanating from each vertex of Σ toward the zeros and poles of x and y , namely toward the points $(0 : 1 : 0)$, $(1 : 0 : 0)$, $(2 : 1 : 0)$, $(-1 : 1 : 0)$. The tropicalization map $\text{trop} : X^{\text{an}} \rightarrow \text{Trop}(X) \subset \mathbf{R}^2$ factors through the retraction map $X^{\text{an}} \rightarrow \Gamma$. See Figure 4.

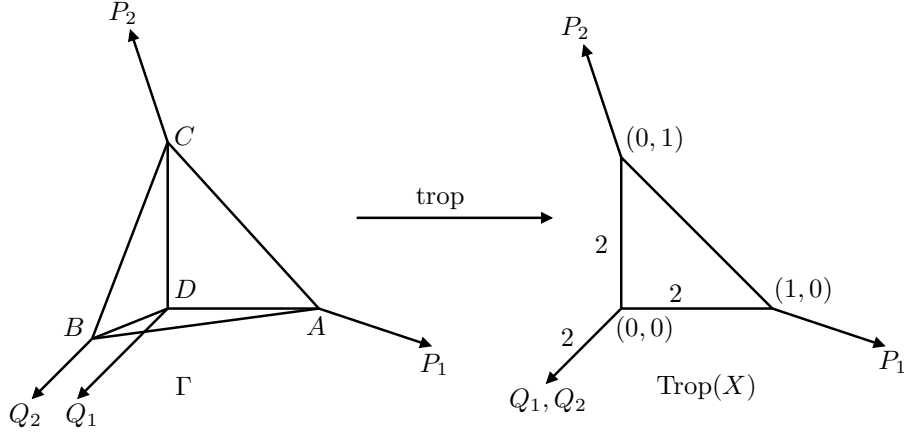


FIGURE 4. The skeleton $\Gamma \subset X^{\text{an}}$ and the tropicalization $\text{Trop}(X)$, where X is the curve from Example 2.9. Here x has divisor $3(P_1) - (P_2) - (Q_1) - (Q_2)$ and y has divisor $3(P_2) - (P_1) - (Q_1) - (Q_2)$ on \hat{X} , where $P_1 = (0 : 1 : 0)$, $P_2 = (1 : 0 : 0)$, $Q_1 = (2 : 1 : 0)$, and $Q_2 = (-1 : 1 : 0)$. The points $A, B, C, D \in \Gamma$ correspond to the irreducible components $x = 0$, $x + y = z$, $y = 0$, and $x - 2y = 2z$, respectively, of \bar{X} . The collapsed segment is \overline{BD} .

The points of Σ corresponding to the irreducible components $x + y = z$ and $x - 2y = 2z$ of \bar{X} , as well as the entire edge of Σ connecting these two points, get mapped by trop to the point $(0, 0) \in \text{Trop}(X)$. This edge therefore has expansion factor zero with respect to trop . The other five bounded edges of $\Sigma \subset \Gamma$ map isometrically (i.e., with expansion factor one) onto their images in $\text{Trop}(X)$. In fact, the tropicalization map is a local isometry everywhere on Γ except along the bounded edge which is contracted to the origin.

Our final example, which is meant to illustrate Theorem 1.4, is a genus 1 curve with multiplicative reduction for which the tropicalization map takes the minimal skeleton isometrically onto its image. This example is also discussed in [KMM09, Example 5.2]).

Example 2.10. Let K be the completion of the field $\mathbf{C}\{\{t\}\}$ of Puiseux series. Consider the curve E' in \mathbf{G}_m^2 cut out by the equation $f(x, y) = x^2y + xy^2 + \frac{1}{t}xy + x + y$. Its closure in \mathbf{P}^2 is the smooth projective genus 1 curve \hat{E}' defined by $\hat{f}(x, y, z) = x^2y + xy^2 + \frac{1}{t}xyz + xz^2 + yz^2$.

Using the description of $\text{Trop}(E')$ as the corner locus of the convex piecewise-linear function associated to f , one sees that $\text{Trop}(E')$ consists of a square with side length 2 plus one ray emanating from each corner of the square; see Figure 5. By restricting f to faces of the Newton complex (see Remark 2.2), one checks that $\text{in}_w(E')$ is reduced and irreducible for every w in $\text{Trop}(E')$. Therefore, by Theorem 1.4 there is a unique graph Γ in the analytification of E' mapping isometrically onto $\text{Trop}(X)$.

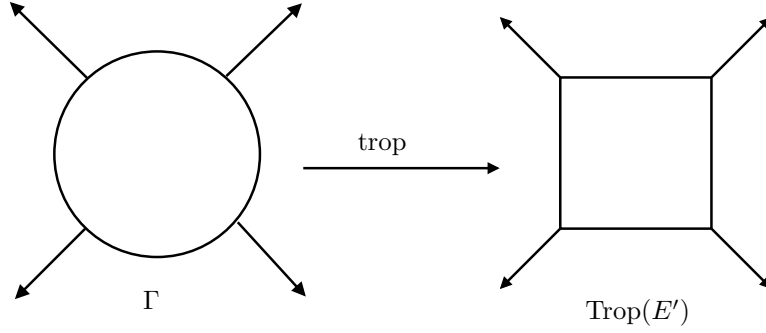


FIGURE 5. The tropicalization of the elliptic curve \widehat{E}' from Example 2.10. The edges in the square each have lattice length 2.

In particular, the analytification of E' contains a loop of length 8. One can check by an explicit computation that $\text{val}(j(\widehat{E}')) = -8$, which is consistent with the fact that the analytification of a smooth projective genus 1 curve is either contractible (if the curve has good reduction) or else contains a unique loop of length $-\text{val}(j)$ (if the curve has multiplicative reduction).

3. ADMISSIBLE ALGEBRAS AND NONARCHIMEDEAN ANALYTIC SPACES

Recall that K is an algebraically closed field that is complete with respect to a nontrivial nonarchimedean valuation $\text{val} : K \rightarrow \mathbf{R} \cup \{\infty\}$. Let $|\cdot| = \exp(-\text{val}(\cdot))$ be the associated absolute value. Let R be the valuation ring of K , let $\mathfrak{m} \subset R$ be its maximal ideal, and let $k = R/\mathfrak{m}$ be its residue field. Choose a nonzero element $\varpi \in \mathfrak{m}$ (as R is not noetherian, it has no uniformizer), so R is ϖ -adically complete. Let $G = \text{val}(K^\times) \subset \mathbf{R}$ be the value group, which is divisible.

3.1. Tate algebras and affinoid algebras. The *Tate algebra* in n variables over K is

$$T_n = K\langle x_1, \dots, x_n \rangle = \left\{ \sum_{I \in \mathbf{Z}_{\geq 0}^n} a_I x^I \in K[[x_1, \dots, x_n]] : |a_I| \rightarrow 0 \text{ as } |I| \rightarrow \infty \right\},$$

where $|(i_1, \dots, i_n)| = i_1 + \dots + i_n$. This noetherian ring enjoys many of the properties satisfied by the polynomial ring $K[x_1, \dots, x_n]$. A (strict) K -affinoid algebra is a K -algebra \mathcal{A} that is isomorphic to a quotient of a Tate algebra. (All K -affinoid algebras in this paper will be assumed to be strict except where explicitly stated otherwise.) If \mathcal{A} is an affinoid algebra we define the *supremum semi-norm* $|\cdot|_{\text{sup}} : \mathcal{A} \rightarrow \mathbf{R}_{\geq 0}$ by

$$|f|_{\text{sup}} = \sup_{\xi \in \text{MaxSpec}(\mathcal{A})} |f(\xi)|.$$

The maximum modulus principle [BGR84, Proposition 6.2.1/4] states that $|f|_{\text{sup}}$ is attained at some point $\xi \in \text{MaxSpec}(\mathcal{A})$, so $|\mathcal{A}|_{\text{sup}} = |K|$. The *ring of power-bounded elements in \mathcal{A}* is the R -algebra $\mathring{\mathcal{A}} = \{f \in \mathcal{A} : |f|_{\text{sup}} \leq 1\}$, the $\mathring{\mathcal{A}}$ -ideal of *topologically nilpotent elements of \mathcal{A}* is $\check{\mathcal{A}} = \{f \in \mathcal{A} : |f|_{\text{sup}} < 1\}$, and the *canonical reduction of \mathcal{A}* is the k -algebra $\tilde{\mathcal{A}} = \mathring{\mathcal{A}}/\check{\mathcal{A}}$. The canonical reduction is always a reduced algebra of finite type over k . It coincides with $\mathring{\mathcal{A}} \otimes_R k$ because the ideal of topologically nilpotent elements is generated by \mathfrak{m} ; this follows from the fact that the value group G is divisible.

The supremum norm on T_n is also called the Gauss norm, and can be calculated as follows: for $f = \sum a_I x^I \in T_n$ we have $|f|_{\text{sup}} = \sup |a_I| = \max |a_I|$. Therefore

$$\mathring{T}_n = R\langle x_1, \dots, x_n \rangle := K\langle x_1, \dots, x_n \rangle \cap R[[x_1, \dots, x_n]] \quad \text{and} \quad \tilde{T}_n = k[x_1, \dots, x_n].$$

We refer the reader to [BGR84, Chapter 6] for details on Tate algebras and affinoid algebras, including the definitions of various classes of analytic domains (e.g. affinoid and rational subdomains) which are mentioned in the sequel.

3.2. Nonarchimedean analytic spaces. Let \mathcal{A} be a K -affinoid algebra, and choose a surjection $\pi : T_n \rightarrow \mathcal{A}$. Then \mathcal{A} inherits a Banach algebra structure by setting $|f|_\pi = \inf\{|g|_{\text{sup}} \mid \pi(g) = f\}$. A *multiplicative seminorm* $\|\cdot\|$ on \mathcal{A} is a map of multiplicative monoids $\|\cdot\| : \mathcal{A} \rightarrow \mathbf{R}$ taking zero to zero and satisfying the triangle inequality, and a seminorm $\|\cdot\|$ is *bounded* if there is some constant C such that $\|f\| \leq C|f|_\pi$ for all f .⁵ The *Berkovich spectrum* $\mathcal{M}(\mathcal{A})$ is defined to be the space of bounded multiplicative seminorms $\|\cdot\|$ on \mathcal{A} extending the absolute value $|\cdot|$ on K , equipped with the coarsest topology making the map $\|\cdot\| \mapsto \|f\|$ continuous for every $f \in \mathcal{A}$. This is exactly the subspace topology induced by the inclusion of $\mathcal{M}(\mathcal{A})$ in $\mathbf{R}^{\mathcal{A}}$.

Following [Ber93], a *quasinet* on a locally compact and locally Hausdorff topological space \mathcal{X} is a collection τ of compact Hausdorff subsets of \mathcal{X} such that (i) each point $x \in \mathcal{X}$ has a neighborhood which is a finite union of elements of τ , and (ii) for each $\mathcal{V} \in \tau$ containing x , the collection $\tau|_{\mathcal{V}} := \{\mathcal{V}' \in \tau \mid \mathcal{V}' \subseteq \mathcal{V}\}$ contains a fundamental system of neighborhoods of x in \mathcal{V} . A quasinet is a *net* if for every $\mathcal{U}, \mathcal{U}' \in \tau$, the collection $\{\mathcal{V} \in \tau \mid \mathcal{V} \subseteq \mathcal{U} \cap \mathcal{U}'\}$ is a quasinet on $\mathcal{U} \cap \mathcal{U}'$. A *K -affinoid atlas* on \mathcal{X} consists of the following data:

- (1) A net τ on \mathcal{X} .
- (2) For each $\mathcal{V} \in \tau$, a K -affinoid algebra $\mathcal{A}_{\mathcal{V}}$ and a homeomorphism $\mathcal{M}(\mathcal{A}_{\mathcal{V}}) \xrightarrow{\sim} \mathcal{V}$.
- (3) For each $\mathcal{V}, \mathcal{V}' \in \tau$ with $\mathcal{V}' \subseteq \mathcal{V}$, a bounded homomorphism of K -Banach algebras $\text{Res}_{\mathcal{V}, \mathcal{V}'} : \mathcal{A}_{\mathcal{V}} \rightarrow \mathcal{A}_{\mathcal{V}'}$ identifying \mathcal{V}' with an affinoid subdomain of \mathcal{V} . (These restriction maps will automatically satisfy the compatibility condition $\text{Res}_{\mathcal{V}, \mathcal{V}''} = \text{Res}_{\mathcal{V}', \mathcal{V}''} \circ \text{Res}_{\mathcal{V}, \mathcal{V}'}$.)

A *K -analytic space* is a locally compact and locally Hausdorff topological space \mathcal{X} together with a K -affinoid atlas. (These are called *strictly* K -analytic spaces in the literature; as all K -analytic spaces we will consider are strict, we refer to them simply as K -analytic spaces.) For example, one can endow $\mathcal{X} = \mathcal{M}(\mathcal{A})$ with a K -affinoid atlas by letting τ be the collection of all affinoid subdomains of \mathcal{X} and assigning to each $\mathcal{V} \in \tau$ the corresponding K -affinoid algebra $\mathcal{A}_{\mathcal{V}}$. Such a space is called *K -affinoid*.

We will not define morphisms between K -analytic spaces here (cf. [Ber93, §1.2] for the precise definition), but the basic idea is that one first defines a strong morphism as a continuous map of topological spaces together with a compatible morphism of K -affinoid atlases and then formally inverts the strong morphisms for which the underlying map of topological spaces is a homeomorphism and each induced morphism of affinoid spaces is the inclusion of an affinoid domain. The resulting category of K -analytic spaces has the following properties:

Facts 3.3.

- (1) The category of K -affinoid spaces is anti-equivalent to the category of K -affinoid algebras, and forms a full subcategory of the category of all K -analytic spaces.
- (2) There is a natural analytification functor from quasi-separated rigid analytic spaces over K (in the sense of [BGR84]) possessing an admissible affinoid covering of finite type to Hausdorff paracompact K -analytic spaces. By Theorem 1.6.1 of [Ber93], this functor gives an equivalence of categories. Under this equivalence, quasi-compact rigid spaces correspond to compact K -analytic spaces.
- (3) A K -analytic space is called *good* if every point has an affinoid neighborhood (as opposed to a neighborhood which is just a finite union of affinoids). The more restrictive notion of K -analytic space defined in [Ber90, §3.1] corresponds to the good K -analytic spaces in the sense of [Ber93].
- (4) There is a natural analytification functor $X \mapsto X^{\text{an}}$ from schemes of finite type over K to good K -analytic spaces. As a topological space, X^{an} can be identified with the set of pairs

⁵The Banach norms on \mathcal{A} induced by any two presentations as quotients of Tate algebras are equivalent [BGR84, Proposition 6.1.3/2], so the boundedness of a seminorm does not depend on the choice of presentation π .

$(x, |\cdot|_x)$ consisting of a point $x \in X$ and an extension of $|\cdot|_K$ to the residue field of x , together with the weakest topology for which U^{an} is open in X^{an} for all Zariski-open subsets $U \subseteq X$ and all functions $U^{\text{an}} \rightarrow \mathbf{R}$ of the form $x \mapsto |f|_x$ for $f \in \mathcal{O}_X(U)$ are continuous. This yields the topological space defined in (2.3) when X is affine.

A covering of a K -analytic space \mathcal{X} by affinoid domains is called *admissible* if its restriction to any affinoid subset $V \in \tau$ is a quasinert on V . A subset U of \mathcal{X} is called a *K -analytic domain* if it has an admissible affinoid covering. A K -analytic domain inherits the structure of a K -analytic space.

3.4. Completed residue fields. If $\mathcal{X} = \mathcal{M}(\mathcal{A})$ is a K -affinoid space (or more generally the spectrum of a Banach ring \mathcal{A}) and $x \in \mathcal{X}$, the kernel of the multiplicative seminorm corresponding to x is a prime ideal \mathfrak{p}_x in \mathcal{A} . The completion of the fraction field of $\mathcal{A}/\mathfrak{p}_x$ is denoted $\mathcal{H}(x)$ and is called the *completed residue field* of x . The image of the homomorphism $\mathcal{A}^\circ \rightarrow \mathcal{H}(x)$ is contained in the valuation ring of $\mathcal{H}(x)$.

For each $f \in \mathcal{A}$ we write $f(x)$ to denote the image of f under the natural map $\mathcal{A} \rightarrow \mathcal{H}(x)$ and $|f(x)|$ to denote the absolute value of $f(x)$ (with respect to the canonical absolute value on $\mathcal{H}(x)$). If x stands for the multiplicative seminorm $\|\cdot\|_x$ then $|f(x)| = \|f\|_x$. The residue field of $\mathcal{H}(x)$ (which we call the *double residue field* of x) is denoted $\widetilde{\mathcal{H}}(x)$. The point x defines a canonical homomorphism $\widetilde{\mathcal{A}} \rightarrow \widetilde{\mathcal{H}}(x)$ whose kernel is a prime ideal $\widetilde{\mathfrak{p}}_x$ of $\widetilde{\mathcal{A}}$.

If \mathcal{X} is any K -analytic space and $\mathcal{V}, \mathcal{V}' \in \tau$ are affinoid subdomains of \mathcal{X} containing a point $x \in \mathcal{X}$, there is a canonical isomorphism between the completed residue fields $\mathcal{H}_{\mathcal{V}}(x)$ and $\mathcal{H}_{\mathcal{V}'}(x)$ (cf. [Ber93, §1.4]). It makes sense, therefore, to speak of the completed residue field $\mathcal{H}(x)$ of x . If \mathcal{X} is a *good* K -analytic space, then there is an associated structure sheaf $\mathcal{O}_{\mathcal{X}}$ for which the field $\mathcal{H}(x)$ is the completion of the residue field of the local ring $\mathcal{O}_{\mathcal{X},x}$ ([Ber93, §2.1]). Little is known about the completed residue field $\mathcal{H}(x)$ when x is a non-good point.

3.5. Admissible R -algebras and Raynaud's generic fiber functor. An R -algebra A is *topologically of finite type* provided that there exists a surjection $\alpha : \hat{T}_n = R\langle x_1, \dots, x_n \rangle \twoheadrightarrow A$ for some n . If we can choose α such that $\ker(\alpha)$ is a finitely generated ideal, we say that A is *topologically of finite presentation*. If in addition A is R -flat, or equivalently if A has no ϖ -torsion, we say that A is an *admissible R -algebra*. By [BL93, Proposition 1.1(b)], a flat R -algebra A of topologically finite type is topologically of finite presentation, and an admissible R -algebra is ϖ -adically complete and separated. We always endow such A with the ϖ -adic topology. If A is an admissible R -algebra then $A_K = A \otimes_R K$ is a K -affinoid algebra and A is contained in the subring A_K° of power-bounded elements.

A formal scheme \mathfrak{X} over $\text{Spf}(R)$ which is locally isomorphic to the formal spectrum of an admissible R -algebra is called an *admissible formal scheme*. All admissible formal schemes appearing in this paper will be assumed to be quasi-compact and separated.

If $\mathfrak{X} = \text{Spf}(A)$ then we write \mathfrak{X}^{an} for the K -affinoid space $\mathcal{M}(A \otimes_R K)$. This construction globalizes to give a functor from admissible formal schemes to analytic spaces that is compatible with fiber products. The image of a formal scheme under this functor is called the *Raynaud generic fiber*.

Remark 3.6. The Raynaud generic fiber \mathfrak{X}^{an} of an admissible formal scheme \mathfrak{X} is not in general a good analytic space. However, \mathfrak{X}^{an} will in fact be a good space in almost all of the cases that we will consider in this paper, namely when \mathfrak{X} is affine (in which case \mathfrak{X}^{an} is by definition affinoid) or when \mathfrak{X} is the completion of a flat and finitely presented R -scheme \mathcal{X} (in which case \mathfrak{X}^{an} is an open analytic domain in the good space $(\mathcal{X}_K)^{\text{an}}$ by [Con99, §A.3]).

An admissible formal scheme \mathfrak{X} with reduced special fiber is called a *formal analytic variety*. If $\text{Spf}(A)$ is a formal affine open subset of a formal analytic variety then A is reduced and A is equal to the full ring of power-bounded elements in $A_K := A \otimes_R K$ by Proposition 3.18. The canonical reduction \hat{A}_K of A_K therefore coincides with $A \otimes_R k$, so if $\text{Spf}(B)$ is a formal affine open subset of $\text{Spf}(A)$ then $\text{Spec}(\hat{B}_K)$ is an affine open subset of $\text{Spec}(\hat{A}_K)$. (Our definition of a formal analytic variety differs from the original one given in [BL85], but the above argument shows that the two definitions are equivalent in our situation.)

Notation. Let \mathfrak{X} be an R -scheme or a formal R -scheme. We denote the special fiber $\mathfrak{X} \otimes_R k$ of \mathfrak{X} by $\overline{\mathfrak{X}}$.

We refer the reader to [BL93] and [Bos05] for a detailed discussion of admissible formal schemes and the Raynaud generic fiber functor.

3.7. Reductions of analytic spaces. If \mathcal{A} is a K -affinoid algebra and $\mathcal{X} = \mathcal{M}(\mathcal{A})$, there is a canonical reduction map $\text{red} : \mathcal{X} \rightarrow \widetilde{\mathcal{X}} := \text{Spec}(\widetilde{\mathcal{A}})$ defined by sending a point x to the prime ideal $\widetilde{\mathfrak{p}}_x$ of $\widetilde{\mathcal{A}}$ (see (3.4)). By [Ber90, Corollary 2.4.2 and Proposition 2.4.4] this map is surjective and anti-continuous, in the sense that the inverse image of an open subset is closed.

Similarly, let $\mathfrak{X} = \text{Spf}(A)$ be an affine admissible formal scheme over $\text{Spf}(R)$. There is a canonical reduction map $\text{red} : \mathfrak{X}^{\text{an}} \rightarrow \overline{\mathfrak{X}}$ defined by

$$\text{red}(\|\cdot\|) = \{a \in A : \|a\| < 1\} \mod \mathfrak{m}_A$$

which coincides with the map defined above when $\mathfrak{X} = \text{Spf}(\mathring{A})$. This construction globalizes to give a reduction map $\text{red} : \mathfrak{X}^{\text{an}} \rightarrow \overline{\mathfrak{X}}$ for any admissible formal scheme \mathfrak{X} . By construction, if $\mathfrak{U} \subset \mathfrak{X}$ is a formal affine open subvariety, then its generic fiber \mathfrak{U}^{an} is the preimage of $\overline{\mathfrak{U}}$ under the reduction map; furthermore, \mathfrak{U}^{an} is an affinoid domain in \mathfrak{X}^{an} , hence is closed. Since any open subset of \mathfrak{X} is covered by finitely many affine opens, this reduction map is again anti-continuous. It is surjective as well: indeed, it suffices to check when $\mathfrak{X} = \text{Spf}(A)$ is affine. The inclusion $A \rightarrow \mathring{A}_K$ induces a finite, surjective morphism $\widetilde{\mathfrak{X}} := \text{Spec}(\widetilde{A}_K) \rightarrow \overline{\mathfrak{X}}$ by Corollary 3.14 (as applied to the identity homomorphism $f : A \rightarrow A$), and it is clear that the triangle

$$\begin{array}{ccc} \mathfrak{X}^{\text{an}} & \xrightarrow{\quad} & \widetilde{\mathfrak{X}} \\ & \searrow \quad \swarrow & \\ & \overline{\mathfrak{X}} & \end{array}$$

is commutative.

In particular, $\overline{\mathfrak{X}}$ is connected if \mathfrak{X}^{an} is connected. The inverse image of a closed point of $\overline{\mathfrak{X}}$ under the reduction map is called a *formal fiber*.

3.8. The Shilov boundary of an analytic space. The *Shilov boundary* of a K -affinoid space $\mathcal{X} = \mathcal{M}(\mathcal{A})$ is defined to be the smallest closed subset $\Gamma(\mathcal{X}) \subset \mathcal{X}$ such that every function $|f|$ for $f \in \mathcal{A}$ attains its maximum at a point of $\Gamma(\mathcal{X})$. Berkovich proves in [Ber90, Proposition 2.4.4] that the Shilov boundary of \mathcal{X} exists and is a finite set. More precisely, he proves that each generic point of $\overline{\mathcal{X}}$ has a unique preimage under red and $\Gamma(\mathcal{X})$ is the inverse image under red of the set of generic points of $\overline{\mathcal{X}}$. By [Ber90, Proposition 2.4.4] or [Thu05, Proposition 2.1.2], the residue field of $\text{red}(\xi)$ is isomorphic to $\widetilde{\mathcal{H}}(\xi)$ for every point $\xi \in \Gamma(\mathcal{X})$. If η is a generic point of $\overline{\mathcal{X}}$, we call $\text{red}^{-1}(\eta) \in \mathcal{X}$ the *Shilov point* associated to η .

The preceding construction also works in the setting of formal analytic varieties. Let \mathfrak{X} be a formal analytic variety over $\text{Spf}(R)$ with Raynaud generic fiber \mathcal{X} . If η is a generic point of $\overline{\mathfrak{X}}$, then there is a unique preimage of η under red which we again call the *Shilov point* x_η associated to η . As above the residue field of η is isomorphic to $\widetilde{\mathcal{H}}(x_\eta)$. The point x_η can be constructed as follows: let $\overline{\mathfrak{U}}$ be an irreducible affine open subscheme of $\overline{\mathfrak{X}}$ with generic point η . Then $\mathcal{U} = \text{red}^{-1}(\overline{\mathfrak{U}})$ is an affinoid domain with irreducible canonical reduction $\overline{\mathfrak{U}}$, so the supremum norm $|\cdot|_{\mathcal{U}}$ on \mathcal{U} is multiplicative. The point $x_\eta = |\cdot|_{\mathcal{U}}$ of \mathcal{X} is the unique point reducing to η .

If \mathcal{X} is a K -affinoid space of pure dimension 1 then $\Gamma(\mathcal{X})$ coincides with the boundary $\partial\mathcal{X}$ in the sense of [Ber90, §2.5.7].

3.9. Analytic curves. Following [Thu05, §2.1.3], we define a (strictly) *analytic curve* over K to be a (good) K -analytic space which is paracompact, of pure dimension 1, and without boundary. The analytification of an algebraic curve over K (by which we mean a one-dimensional separated integral scheme of finite type over K) is always an analytic curve in this sense.

If \mathcal{X} is an analytic curve and $\mathcal{V} \subset \mathcal{X}$ is an affinoid domain, then by [Ber90, Proposition 3.1.3] and [Thu05, Proposition 2.1.12] the following three (finite) sets coincide: (i) the topological boundary $\partial_{\text{top}} \mathcal{V}$ of \mathcal{V} in \mathcal{X} ; (ii) the boundary $\partial \mathcal{V}$ of \mathcal{V} in the sense of [Ber90, §2.5.7]; and (iii) the Shilov boundary $\Gamma(\mathcal{V})$ of \mathcal{V} .

3.10. Types of points in an analytic curve. Let x be a point in a K -analytic curve \mathcal{X} and let $\mathcal{H}(x)$ be its completed residue field (3.4). The extension $\widetilde{\mathcal{H}(x)}/k$ has transcendence degree $s(x) \leq 1$ and the abelian group $|\mathcal{H}(x)^\times|/|K^\times|$ has rank $t(x) \leq 1$. Moreover, the integers $s(x)$ and $t(x)$ must satisfy the *Abhyankar inequality* [Vaq00, Theorem 9.2]

$$s(x) + t(x) \leq 1.$$

Using the terminology from [Ber90] and [Ber93] (see also [Thu05, §2.1]), we say that x is *type-2* if $s(x) = 1$ and *type-3* if $t(x) = 1$. If $s(x) = t(x) = 0$, then x is called *type-1* if $\mathcal{H}(x) = K$ and *type-4* otherwise. Points of type 4 will not play any significant role in this paper. We define

$$\begin{aligned} \mathbf{H}_o(\mathcal{X}) &= \{\text{all points of } \mathcal{X} \text{ of types 2 and 3}\} \\ \mathbf{H}(\mathcal{X}) &= \{\text{all points of } \mathcal{X} \text{ of types 2, 3, and 4}\}. \end{aligned}$$

We call $\mathbf{H}_o(\mathcal{X})$ the set of *skeletal points*, because it is the union of all skeleta of admissible formal models of \mathcal{X} (Corollary 5.56), and $\mathbf{H}(\mathcal{X})$ the set of *norm points* of \mathcal{X} , because it is the set of all points corresponding to norms on the function field $K(X)$ that extend the given norm on K . If $\mathcal{X} = X^{\text{an}}$ is the analytification of an algebraic curve X over K , then $X(K) \subset X^{\text{an}}$ is naturally identified with the set of type-1 points of X^{an} , so $\mathbf{H}(X^{\text{an}}) = X^{\text{an}} \setminus X(K)$. (Recall that we are assuming throughout this discussion that K is algebraically closed.)

When $X = \mathbf{A}_K^1$, the classification of points in X^{an} has a particularly simple interpretation (cf. [Ber90, §1] and [BR10, §2.1]). If $D = \{z \in K : |z - a| \leq r\}$ is a (possibly degenerate) closed ball in K , then the supremum norm $\|f\|_D = \sup_{z \in D} |f(z)|$ is a multiplicative seminorm on the polynomial ring $K[T]$ extending the absolute value on K ; it thus gives rise to a point ζ_D of \mathbf{A}_{an}^1 . The point ζ_D is type-1 if $r = 0$, type-2 if $r \in |K^\times|$, and type-3 if $r \notin |K|$. More generally, every nested sequence of closed balls $D_1 \supseteq D_2 \supseteq \cdots$ gives rise to a point of \mathbf{A}_{an}^1 since $\lim_{n \rightarrow \infty} \|\cdot\|_{D_n}$ is also a multiplicative seminorm. According to *Berkovich's classification theorem*, every point of \mathbf{A}_{an}^1 arises in this way. A point of \mathbf{A}_{an}^1 corresponding to a nested sequence $D_1 \supseteq D_2 \supseteq \cdots$ is of type 4 if and only if $\bigcap_{n \geq 1} D_n = \emptyset$. The space \mathbf{P}_{an}^1 is the one point compactification of \mathbf{A}_{an}^1 , obtained by adjoining a single type-1 point ∞ .

3.11. Some facts about admissible R -algebras. Here we collect some more or less well-known facts about admissible R -algebras that will be needed in the sequel.

Proposition 3.12.

- (1) If A is a finitely presented and flat R -algebra then its ϖ -adic completion \widehat{A} is an admissible R -algebra.
- (2) If $f : A \twoheadrightarrow B$ is a surjective homomorphism of finitely presented and flat R -algebras with kernel \mathfrak{a} then $\widehat{f} : \widehat{A} \twoheadrightarrow \widehat{B}$ is a surjection of admissible R -algebras with kernel $\mathfrak{a}\widehat{A}$.

Proof. In the situation of (2), for $n \geq 0$ we have a Mittag-Leffler inverse system of exact sequences

$$0 \longrightarrow \mathfrak{a}/\varpi^n \mathfrak{a} \longrightarrow A/\varpi^n A \longrightarrow B/\varpi^n B \longrightarrow 0$$

since B is R -flat. Therefore the inverse limit $0 \rightarrow \widehat{\mathfrak{a}} \rightarrow \widehat{A} \rightarrow \widehat{B} \rightarrow 0$ is exact, which proves the surjectivity statement (see also [Ull95, Lemma 1.4]). Next we prove (1). The above argument as applied to a presentation of A shows that \widehat{A} is topologically finitely generated, so we only need to show that \widehat{A} is ϖ -torsion-free. Let $x = (x_m \bmod \varpi^m A) \in \varprojlim_m A/\varpi^m A$ be nonzero, so $x_n \notin \varpi^n A$ for some n . Then $x_{n+1} \notin \varpi^n A$, so $\varpi x_{n+1} \notin \varpi^{n+1} A$ since A is ϖ -torsion-free, so $\varpi x = (\varpi x_m \bmod \varpi^m A) \neq 0$. Therefore \widehat{A} is admissible.

It remains to prove that $\hat{\mathfrak{a}} = \ker(\hat{f}) = \mathfrak{a}\hat{A}$ in (2). Since A and B are finitely presented, \mathfrak{a} is finitely generated, so $\mathfrak{a}\hat{A}$ is finitely generated, hence closed in \hat{A} by [BL93, Lemma 1.2(b)]. The ϖ -adic topology on \hat{A} restricts to the ϖ -adic topology on $\hat{\mathfrak{a}}$ by Lemma 1.2(a) of loc. cit., so $\mathfrak{a}\hat{A}$ is ϖ -adically closed in $\hat{\mathfrak{a}}$. Since \mathfrak{a} is ϖ -adically dense in $\hat{\mathfrak{a}}$ it follows that $\hat{\mathfrak{a}} = \mathfrak{a}\hat{A}$. ■

We set the following notation, which we will use until (3.20): A and B will denote admissible R -algebras, $\overline{A} = A \otimes_R k$ and $\overline{B} = B \otimes_R k$ their reductions, and $A_K = A \otimes_R K$ and $B_K = B \otimes_R K$ the associated K -affinoid algebras. We let $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, $\overline{\mathfrak{X}} = \mathrm{Spec}(\overline{A})$, and $\overline{\mathfrak{Y}} = \mathrm{Spec}(\overline{B})$. Let $f : A \rightarrow B$ be a homomorphism, let $\overline{f} : \overline{A} \rightarrow \overline{B}$ and $f_K : A_K \rightarrow B_K$ be the induced homomorphisms, and let $\varphi : \mathfrak{Y} \rightarrow \mathfrak{X}$ and $\overline{\varphi} : \overline{\mathfrak{Y}} \rightarrow \overline{\mathfrak{X}}$ be the induced morphisms.

Proposition 3.13.

- (1) f is flat if and only if $\overline{f} : \overline{A} \rightarrow \overline{B}$ is flat.
- (2) f is finite if and only if $f_K : A_K \rightarrow B_K$ is finite.

Proof. The ‘only if’ directions are clear. Suppose that \overline{f} is flat. By [BL93, Lemma 1.6], it suffices to show that $f_n : A_n \rightarrow B_n$ is flat for all $n \geq 0$, where $A_n = A/\varpi^{n+1}A$ and $B_n = B/\varpi^{n+1}B$. But A_n and B_n are of finite presentation and flat over $R_n = R/\varpi^{n+1}R$, so f_n is flat by the fibral flatness criterion [EGAIV₃, Corollaire 11.3.1].

Now suppose that f_K is finite. Choose a surjection $\hat{T}_n \twoheadrightarrow A$. The induced homomorphism $T_n \rightarrow A \otimes_R K \rightarrow B \otimes_R K$ is finite, so by [BGR84, Theorem 6.3.5/1] the composition $\hat{T}_n \twoheadrightarrow A \rightarrow (B \otimes_R K)^\circ$ is integral. Hence $A \rightarrow (B \otimes_R K)^\circ$ is integral, so $A \rightarrow B$ is integral since $B \subset (B \otimes_R K)^\circ$. Then $f_n : A/\varpi^{n+1}A \rightarrow B/\varpi^{n+1}B$ is of finite type and integral for all $n \geq 0$, so f_n is finite, so f is finite by [BL93, Lemma 1.5]. ■

Corollary 3.14. *Suppose that $f_K : A_K \rightarrow B_K$ is finite and dominant, i.e., that $\ker(f_K)$ is nilpotent. Then $\overline{f} : \overline{A} \rightarrow \overline{B}$ is finite and $\overline{\varphi} : \overline{\mathfrak{Y}} \rightarrow \overline{\mathfrak{X}}$ is surjective.*

Proof. Since $A \subset A_K$ and $B \subset B_K$ we have that $\ker(f)$ is nilpotent, and f is finite by Proposition 3.13. Hence $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ is surjective, so $\overline{\mathfrak{Y}} \rightarrow \overline{\mathfrak{X}}$ is surjective. Finiteness of f implies finiteness of \overline{f} . ■

We say that a ring is *equidimensional of dimension d* provided that every maximal ideal has height d . Let \mathcal{A} be a K -affinoid algebra, and let $\mathcal{X} = \mathcal{M}(\mathcal{A})$. Then \mathcal{A} is equidimensional of dimension d if and only if $\dim(\mathcal{O}_{\mathcal{X},x}) = d$ for every $x \in \mathrm{MaxSpec}(\mathcal{A})$ by [BGR84, Proposition 7.3.2/8]. In particular, if $\mathcal{M}(\mathcal{B})$ is an affinoid domain in $\mathcal{M}(\mathcal{A})$ and \mathcal{A} is equidimensional of dimension d then so is \mathcal{B} .

Proposition 3.15. *If A_K is equidimensional of dimension d then \overline{A} is equidimensional of dimension d .*

Proof. Replacing \mathfrak{X} with an irreducible formal affine open subset, we may assume that \mathfrak{X} is irreducible. Let $R\langle x_1, \dots, x_n \rangle \twoheadrightarrow A$ be a presentation of A . By Noether normalization [BGR84, Theorem 6.1.2/1] we can choose the x_i such that $K\langle x_1, \dots, x_d \rangle \rightarrow A_K$ is finite and injective, where $d = \dim(A_K)$. Then $\overline{\mathfrak{X}} \rightarrow \mathbf{A}_k^d$ is finite and surjective by Corollary 3.14. ■

Corollary 3.16. *Suppose that $f_K : A_K \rightarrow B_K$ is finite and dominant, and that A_K and B_K are equidimensional (necessarily of the same dimension). Then $\overline{\varphi} : \overline{\mathfrak{Y}} \rightarrow \overline{\mathfrak{X}}$ is finite and surjective, and the image of an irreducible component of $\overline{\mathfrak{Y}}$ is an irreducible component of $\overline{\mathfrak{X}}$.*

Proof. This follows immediately from Proposition 3.15 and Corollary 3.14. ■

The following theorem uses the fact that K is algebraically closed in an essential way. It can be found in [BL85, Proposition 1.1].

Theorem 3.17. *Let \mathcal{A} be a K -affinoid algebra. Then $\hat{\mathcal{A}}$ is admissible if and only if \mathcal{A} is reduced.*

Proof. Since $\hat{\mathcal{A}}$ is always R -flat, by [BL93, Proposition 1.1(c)] the issue is whether $\hat{\mathcal{A}}$ is topologically finitely generated. Suppose that \mathcal{A} is reduced. By [BGR84, Theorem 6.4.3/1] there is a surjection $T_n \twoheadrightarrow \mathcal{A}$ such that the residue norm on \mathcal{A} agrees with the supremum norm; then by Proposition 6.4.3/3(i) of loc. cit. the induced homomorphism $\hat{T}_n \rightarrow \hat{\mathcal{A}}$ is surjective. The converse follows in a similar way from Theorem 6.4.3/1 and Corollary 6.4.3/6 of loc. cit. ■

Proposition 3.18. *The ring \overline{A} is reduced if and only if $A = \mathring{A}_K$, in which case A is reduced.*

Proof. If $A = \mathring{A}_K$ then A is reduced by Theorem 3.17, so $\overline{A} = \widetilde{A}_K$ is reduced. Conversely, suppose that \overline{A} is reduced. Let $\alpha : \mathring{T}_n \twoheadrightarrow A$ be a surjection. Since the \mathring{T}_n -ideal $\mathring{T}_n + \ker(\alpha) = \mathfrak{m}\mathring{T}_n + \ker(\alpha)$ is the kernel of the composite homomorphism $\mathring{T}_n \rightarrow A \rightarrow \overline{A}$, it is a reduced ideal; hence by [BGR84, Propositions 6.4.3/4, 6.4.3/3(i)] we have $A = \alpha(\mathring{T}_n) = \mathring{A}_K$. ■

Corollary 3.19. *If \overline{A} is an integral domain then A_K is an integral domain and $|\cdot|_{\text{sup}}$ is multiplicative.*

Proof. By Proposition 3.18 we have $\widetilde{A}_K = \overline{A}$, so the result follows from [BGR84, Proposition 6.2.3/5]. ■

3.20. Finite morphisms of pure degree. In general there is not a good notion of the ‘degree’ of a finite morphism $Y \rightarrow X$ between noetherian schemes when X is not irreducible, since the degree of the induced map on an irreducible component of X can vary from component to component. The notion of a morphism having ‘pure degree’ essentially means that the degree is the same on every irreducible component of X . This notion is quite well behaved in that it respects analytification of algebraic varieties and of admissible formal schemes. The definition of a morphism of pure degree is best formulated in the language of fundamental cycles, so we will give a brief review of the theory of cycles on noetherian schemes which are not necessarily of finite type over a field. A good reference for this material is [Tho90].

3.21. Let X be a noetherian scheme. A *cycle* on X is a finite formal sum $\sum_W n_W \cdot W$, where $n_W \in \mathbb{Z}$ and W ranges over the irreducible closed subsets of X . The group of cycles on X is denoted $C(X)$. The *fundamental cycle* of X is the cycle

$$[X] = \sum_{\zeta} \text{length}_{\mathcal{O}_{X,\zeta}}(\mathcal{O}_{X,\zeta}) \cdot \overline{\{\zeta\}},$$

where the sum is taken over all generic points of X . Let $f : Y \rightarrow X$ be a morphism of noetherian schemes, let $W \subset Y$ be an irreducible closed subset, and let ζ be the generic point of W . Let $\deg_{\zeta}(f)$ denote the degree of the extension of residue fields $[\kappa(\zeta) : \kappa(f(\zeta))]$ if this quantity is finite, and set $\deg_{\zeta}(f) = 0$ otherwise. We define

$$f_*(W) = \deg_{\zeta}(f) \cdot \overline{f(W)} \in C(X).$$

Extending linearly, we obtain a *pushforward homomorphism* $f_* : C(Y) \rightarrow C(X)$. If V is an irreducible closed subset of X with generic point ξ , we define

$$f^*(V) = \sum_{\eta \mapsto \xi} \text{length}_{\mathcal{O}_{f^{-1}\xi,\eta}}(\mathcal{O}_{f^{-1}\xi,\eta}) \cdot \overline{\{\eta\}} \in C(Y),$$

where the sum is taken over all generic points η of $f^{-1}(\xi)$. Extending linearly yields a *pullback homomorphism* $f^* : C(X) \rightarrow C(Y)$.

Here we collect some standard facts about cycles on noetherian schemes. See [Tho90, Lemmas 2.4, 2.5, and 4.8] for the proofs.

Proposition 3.22. *Let $f : Y \rightarrow X$ be a morphism of noetherian schemes.*

- (1) *If $g : X \rightarrow Z$ is another morphism of noetherian schemes, then $(g \circ f)_* = g_* \circ f_*$.*
- (2) *Let $g : X' \rightarrow X$ be a morphism of noetherian schemes, let $Y' = X' \times_X Y$, let $f' : Y' \rightarrow X'$ and $h : Y' \rightarrow Y$ be the projections, and suppose that either f or g satisfies the property that all of its induced residue field extensions are finite. Then*

$$g^* f_* = f'_* h^* \quad \text{as maps } C(Y) \longrightarrow C(X').$$

- (3) *If f is flat and $V \subset X$ is a closed subscheme then*

$$f^*[V] = [f^{-1}V] \in C(Y).$$

- (4) *If f is flat and $g : X \rightarrow Z$ is a morphism of noetherian schemes then $(g \circ f)^* = f^* \circ g^*$.*

Definition 3.23. Let $f : Y \rightarrow X$ be a finite morphism of noetherian schemes. We say that f has *pure degree* δ and we write $[Y : X] = \delta$ provided that $f_*[Y] = \delta[X]$; here $\delta \in \mathbf{Q}$ need not be an integer.

Remark 3.24. Let $f : Y \rightarrow X$ be a finite morphism of noetherian schemes.

- (1) If X is irreducible and every generic point of Y maps to the generic point of X , then f automatically has a pure degree, which we simply call the *degree* of f . Moreover, if X is integral with generic point ζ then the degree of f is the dimension of $\Gamma(f^{-1}(\zeta), \mathcal{O}_{f^{-1}(\zeta)})$ as a vector space over the function field $\mathcal{O}_{X,\zeta}$. In particular, if f is a finite and dominant morphism of integral schemes, then the (pure) degree of f is the degree of the extension of function fields.
- (2) Let ζ be a generic point of X and let $C = \overline{\{\zeta\}}$ be the corresponding irreducible component. Define the *multiplicity of C in X* to be the quantity

$$\text{mult}_X(C) = \text{length}_{\mathcal{O}_{X,\zeta}}(\mathcal{O}_{X,\zeta}),$$

so $[X] = \sum_C \text{mult}_X(C) \cdot C$. It follows that f has pure degree δ if and only if (1) every irreducible component D of Y maps to an irreducible component of X , and (2) for every irreducible component C of X we have

$$(3.24.1) \quad \delta \text{mult}_X(C) = \sum_{D \rightarrow C} \text{mult}_Y(D) [D : C],$$

where $[D : C]$ is the usual degree of a finite morphism of integral schemes.

- (3) Let $g : X \rightarrow Z$ be another finite morphisms of noetherian schemes. Suppose that f has pure degree δ and g has pure degree ε . Then $g \circ f$ has pure degree $\delta\varepsilon$ by Proposition 3.22(1).

The following examples are meant to illustrate the generality in which we are working.

Example 3.25.

- (1) Let $X = \text{Spec}(k[x, x^{-1}, y]/(y^2))$ and $Y = \text{Spec}(k[T, T^{-1}])$, and define $f : Y \rightarrow X$ by $x \mapsto T^2$ and $y \mapsto 0$. The local ring at the generic point of X is $k(x)[y]/(y^2)$ and the local ring at the generic point of Y is $k(T)$, so $[X] = 2 \cdot X$ and $[Y] = Y$. Since $[k(T) : k(x)] = 2$ we have

$$f_*[Y] = f_*Y = 2 \cdot X = [X],$$

so f has pure degree 1.

- (2) Let $X = \text{Spec}(\mathbf{Q}[x, y]/(y^2))$, let $Y = \text{Spec}(\mathbf{Q}[x]) \cong X^{\text{red}}$, and let $f : Y \hookrightarrow X$ be the canonical closed immersion. The local ring at the generic point of X is isomorphic to the ring of dual numbers $\mathbf{Q}(x)[y]/(y^2)$ over $\mathbf{Q}(x)$, and the local ring at the generic point of Y is $\mathbf{Q}(x)$. Hence $[X] = 2 \cdot X$ and $[Y] = Y$; since all residue field extensions of f are trivial, we have

$$f_*[Y] = f_*Y = X = \frac{1}{2}[X],$$

so $[Y : X] = 1/2$.

Proposition 3.26. Let X, Y, X' be noetherian schemes, let $f : Y \rightarrow X$ be a finite morphism, let $g : X' \rightarrow X$ be a flat morphism, let $Y' = Y \times_X X'$, and let $f' : Y' \rightarrow X'$ be the projection.

- (1) If f has pure degree δ then f' has pure degree δ .
- (2) If g is surjective then f has pure degree δ if and only if f' has pure degree δ .

Proof. Let $h : Y' \rightarrow Y$ be the other projection, so h is flat. By Proposition 3.22(2,3) we have

$$f'_*[Y'] = f'_*h^*[Y] = g^*f_*[Y] = \delta g^*[X] = \delta[X'],$$

which proves (1). Conversely, suppose that g is surjective (and flat) and that $f'_*[Y'] = \delta[X']$. Then

$$g^*f_*[Y] = f'_*h^*[Y] = f'_*[Y'] = \delta[X'] = g^*(\delta[X]),$$

so we are done because g^* is visibly injective in this situation. ■

3.27. Next we will define pure-degree morphisms of analytic spaces. As above, we must first review the notion of the fundamental cycle of an analytic space, as defined by Gubler [Gub98, §2]. As Gubler uses the language of rigid-analytic spaces, we make some remarks about the relation between rigid spaces and K -analytic spaces.

There are natural notions of closed immersions and of finite, proper, and flat morphisms of K -analytic spaces; see [Ber93, §1.3-1.5]. If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of rigid-analytic spaces as in Fact 3.3(2) and $f^{\text{an}} : \mathcal{X}^{\text{an}} \rightarrow \mathcal{Y}^{\text{an}}$ is the induced morphism of K -analytic spaces, then f is a closed immersion (resp. finite, proper, flat morphism) if and only if f^{an} satisfies the same property. For proper morphisms this is a difficult fact proved by Temkin [Tem00, Corollary 4.5], and for flat morphisms the proof is again difficult, and is due to Ducros [Duc11, Corollary 7.2]. Closed immersions are finite, finite morphisms are proper, and the inclusion of an affinoid domain into an analytic space is flat.

Let \mathcal{X} be a K -analytic space (assumed from now on to be Hausdorff and paracompact). A Zariski-closed subspace of \mathcal{X} is by definition an isomorphism class of closed immersions $\mathcal{V} \hookrightarrow \mathcal{X}$. A Zariski-closed subspace of \mathcal{X} is *irreducible* if it cannot be expressed as a union of two proper Zariski-closed subspaces. Gubler [Gub98, §2] defines a *cycle* on \mathcal{X} to be a locally finite formal sum $\sum_{\mathcal{V}} n_{\mathcal{V}} \mathcal{V}$, where $n_{\mathcal{V}} \in \mathbb{Z}$ and \mathcal{V} ranges over the irreducible Zariski-closed subspaces of \mathcal{X} ; ‘locally finite’ means that there exists an admissible covering of \mathcal{X} by affinoid domains intersecting only finitely many \mathcal{V} with $n_{\mathcal{V}} \neq 0$. Let $C(\mathcal{X})$ denote the group of cycles on \mathcal{X} .

3.27.1. If $\mathcal{X} = \mathcal{M}(A)$ is affinoid then the Zariski-closed subspaces of \mathcal{X} are in natural inclusion-reversing bijection with the ideals of A ; therefore we have an identification $C(\mathcal{X}) = C(\text{Spec}(A))$, which we will make implicitly from now on.

3.27.2. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of K -analytic spaces. If f is proper then there is a pushforward homomorphism $f_* : C(\mathcal{Y}) \rightarrow C(\mathcal{X})$, and if f is flat then there is a pullback homomorphism $f^* : C(\mathcal{X}) \rightarrow C(\mathcal{Y})$. There is a canonical *fundamental cycle* $[\mathcal{X}] \in C(\mathcal{X})$ which is uniquely determined by the property that for every affinoid domain $\iota : \mathcal{M}(A) \hookrightarrow \mathcal{X}$, we have $\iota^*[\mathcal{X}] = [\mathcal{M}(A)] = [\text{Spec}(A)]$. The analogue of Proposition 3.22 holds in this situation: see [Gub98, 2.6, 2.7, 2.8, and Proposition 2.12].

Definition 3.28. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a finite morphism of K -analytic spaces. We say that f has *pure degree* δ and we write $[\mathcal{Y} : \mathcal{X}] = \delta$ provided that $f_*[\mathcal{Y}] = \delta[\mathcal{X}]$. Again $\delta \in \mathbb{Q}$ need not be an integer.

Remark 3.29. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a finite morphism of K -analytic spaces.

- (1) If $\mathcal{X} = \mathcal{M}(A)$ and $\mathcal{Y} = \mathcal{M}(B)$ are affinoid then $f : \mathcal{M}(B) \rightarrow \mathcal{M}(A)$ has pure degree δ if and only if the map of affine schemes $\text{Spec}(B) \rightarrow \text{Spec}(A)$ has pure degree δ .
- (2) If f has pure degree δ and $g : \mathcal{X} \rightarrow \mathcal{Z}$ is a finite morphism of analytic spaces of pure degree ε then $g \circ f$ has pure degree $\delta\varepsilon$.

Proposition 3.30. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a finite morphism of K -analytic spaces.

- (1) If f has pure degree δ , $\mathcal{M}(A) \subset \mathcal{X}$ is an affinoid domain, and $\mathcal{M}(B) = f^{-1}(\mathcal{M}(A))$, then $\mathcal{M}(B) \rightarrow \mathcal{M}(A)$ has pure degree δ .
- (2) If there exists an admissible cover $\mathcal{X} = \bigcup_i \mathcal{M}(A_i)$ of \mathcal{X} by affinoid domains such that $\mathcal{M}(B_i) = f^{-1}(\mathcal{M}(A_i)) \rightarrow \mathcal{M}(A_i)$ has pure degree δ for each i , then f has pure degree δ .

Proof. Since the inclusion $\mathcal{M}(A) \hookrightarrow \mathcal{X}$ is flat, the first part follows as in the proof of Proposition 3.26(1). In the situation of (2), let $f_i = f|_{\mathcal{M}(B_i)} : \mathcal{M}(B_i) \rightarrow \mathcal{M}(A_i)$, and assume that $(f_i)_*[\mathcal{M}(B_i)] = \delta[\mathcal{M}(A_i)]$ for all i . Arguing as in the proof of Proposition 3.26(1), we see that the pullback of $f_*[\mathcal{Y}]$ to $\mathcal{M}(A_i)$ is equal to $\delta[\mathcal{M}(A_i)]$ for all i ; since $[\mathcal{X}]$ is the unique cycle which pulls back to $[\mathcal{M}(A_i)]$ for all i , this shows that $f_*[\mathcal{Y}] = \delta[\mathcal{X}]$. ■

The property of being a finite morphism of pure degree is compatible with analytification:

Proposition 3.31. Let $f : Y \rightarrow X$ be a morphism of finite-type K -schemes. Then f is finite of pure degree δ if and only if $f^{\text{an}} : Y^{\text{an}} \rightarrow X^{\text{an}}$ is finite of pure degree δ .

Proof. By [Con99, Theorem A.2.1], f is finite if and only if f^{an} is finite. Hence we may assume that $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine. If $\mathcal{M}(\mathcal{A}) \subset X^{\text{an}}$ is an affinoid domain then $\text{Spec}(\mathcal{A}) \rightarrow \text{Spec}(A)$ is flat by Lemma A.1.2 of loc. cit., and if $\{\mathcal{M}(\mathcal{A}_i)\}_{i \in I}$ is an admissible covering of X^{an} then $\coprod_{i \in I} \text{Spec}(\mathcal{A}_i) \rightarrow \text{Spec}(A)$ is flat and surjective. Let $\mathcal{M}(\mathcal{B}_i) = f^{-1}(\mathcal{M}(\mathcal{A}_i))$. We claim that $\mathcal{B}_i = B \otimes_A \mathcal{A}_i$. Since $B \otimes_A \mathcal{A}_i$ is finite over \mathcal{A}_i it is affinoid by [BGR84, Proposition 6.1.1/6], so the claim follows easily from the universal property of the analytification (see also [Con99, §A.2]). Hence by Proposition 3.26(2), f has pure degree δ if and only if $\text{Spec}(\mathcal{B}_i) \rightarrow \text{Spec}(\mathcal{A}_i)$ has pure degree δ for each i ; by Remark 3.29(1), this is the case if and only if $\mathcal{M}(\mathcal{B}_i) \rightarrow \mathcal{M}(\mathcal{A}_i)$ has pure degree δ for each i , which is equivalent to f^{an} having pure degree δ by Proposition 3.30(2). ■

The following counterpart to Proposition 3.31 allows us to compare the degrees of the generic and special fibers of a finite morphism of admissible formal schemes. It will play a key role throughout this paper.

Proposition 3.32. (Projection formula) *Let $f : \mathfrak{Y} \rightarrow \mathfrak{X}$ be a finite morphism of admissible formal schemes, and let $f^{\text{an}} : \mathfrak{Y}^{\text{an}} \rightarrow \mathfrak{X}^{\text{an}}$ and $\bar{f} : \bar{\mathfrak{Y}} \rightarrow \bar{\mathfrak{X}}$ be the induced morphisms on the generic and special fibers, respectively. If f^{an} has pure degree δ then \bar{f} has pure degree δ .*

Proof. The theory of cycles on analytic spaces discussed above is part of Gubler's more general intersection theory on admissible formal schemes, and our 'projection formula' is in fact a special case of Gubler's projection formula [Gub98, Proposition 4.5]; this can be seen as follows. Choose any $\varpi \in K^\times$ with $\text{val}(\varpi) \in (0, \infty)$, and let D be the Cartier divisor on \mathfrak{X} defined by ϖ . Essentially by definition (cf. (3.8) and (3.10) of loc. cit.) the intersection product $D \cdot [\mathfrak{X}^{\text{an}}]$ is equal to $\text{val}(\varpi) [\bar{\mathfrak{X}}]$, and likewise $(f^*D) \cdot [\mathfrak{Y}^{\text{an}}] = \text{val}(\varpi) [\bar{\mathfrak{Y}}]$. Hence if $f^{\text{an}}[\mathfrak{Y}^{\text{an}}] = \delta [\mathfrak{X}^{\text{an}}]$ then

$$\text{val}(\varpi) \bar{f}_*[\bar{\mathfrak{Y}}] = f_*((f^*D) \cdot [\mathfrak{Y}^{\text{an}}]) = D \cdot f_*[\mathfrak{Y}^{\text{an}}] = D \cdot (\delta [\mathfrak{X}^{\text{an}}]) = \text{val}(\varpi) \delta [\bar{\mathfrak{X}}],$$

where the second equality is by Gubler's projection formula. Canceling the factors of $\text{val}(\varpi)$ yields Proposition 3.32. ■

Remark 3.33. The converse to Proposition 3.32 does not hold in general. The following example is due to Gubler: let $\mathfrak{X} = \text{Spf}(R[x]/(x(x - \varpi)))$ and $\mathfrak{Y} = \mathfrak{X} \amalg \text{Spf}(R)$, and let $f : \mathfrak{Y} \rightarrow \mathfrak{X}$ be the map which is the identity on \mathfrak{X} and which maps $\text{Spf}(R)$ to \mathfrak{X} via $x \mapsto 0$. Then f^{an} does not have a pure degree, but \bar{f} does since $\bar{\mathfrak{X}}$ is a point.

3.34. Here we note some special cases of the projection formula:

- (1) Suppose that $\mathfrak{X} = \text{Spf}(A)$ and $\mathfrak{Y} = \text{Spf}(B)$, and that A is an integral domain with fraction field Q . If all generic points of $\text{Spec}(B \otimes_R K)$ map to the generic point of $\text{Spec}(A \otimes_R K)$ then $\mathcal{M}(B \otimes_R K) \rightarrow \mathcal{M}(A \otimes_R K)$ is finite with pure degree equal to $\dim_Q(B \otimes_A Q)$. By (3.24.1), for every irreducible component $\bar{\mathfrak{C}}$ of $\bar{\mathfrak{X}}$ we have

$$(3.34.1) \quad \dim_Q(B \otimes_A Q) \cdot \text{mult}_{\bar{\mathfrak{X}}}(\bar{\mathfrak{C}}) = \sum_{\bar{\mathfrak{D}} \rightarrow \bar{\mathfrak{C}}} \text{mult}_{\bar{\mathfrak{Y}}}(\bar{\mathfrak{D}}) \cdot [\bar{\mathfrak{D}} : \bar{\mathfrak{C}}],$$

where the sum is taken over all irreducible components $\bar{\mathfrak{D}}$ of $\bar{\mathfrak{Y}}$ that surject onto $\bar{\mathfrak{C}}$.

- (2) Suppose that $f^{\text{an}} : \mathfrak{Y}^{\text{an}} \rightarrow \mathfrak{X}^{\text{an}}$ is an isomorphism. Then for every irreducible component $\bar{\mathfrak{C}}$ of $\bar{\mathfrak{X}}$ we have

$$(3.34.2) \quad \text{mult}_{\bar{\mathfrak{X}}}(\bar{\mathfrak{C}}) = \sum_{\bar{\mathfrak{D}} \rightarrow \bar{\mathfrak{C}}} \text{mult}_{\bar{\mathfrak{Y}}}(\bar{\mathfrak{D}}) \cdot [\bar{\mathfrak{D}} : \bar{\mathfrak{C}}],$$

where the sum is taken over all irreducible components $\bar{\mathfrak{D}}$ of $\bar{\mathfrak{Y}}$ that surject onto $\bar{\mathfrak{C}}$, because an isomorphism has pure degree 1.

4. TROPICAL INTEGRAL MODELS

We continue to assume that K is an algebraically closed field which is complete with respect to a nontrivial nonarchimedean valuation.

Notation 4.1. Let $M \cong \mathbf{Z}^n$ be a lattice, with dual lattice $N = \text{Hom}(M, \mathbf{Z})$. If H is an additive subgroup of \mathbf{R} , we write M_H for $M \otimes_{\mathbf{Z}} H$, so N_H is naturally identified with $\text{Hom}(M, H)$. We write $\langle \cdot, \cdot \rangle$ to denote the canonical pairings $M \times N \rightarrow \mathbf{Z}$ and $M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$.

Let $\mathbf{T} = \text{Spec } K[M]$ be the torus over K with character lattice M . For u in M , we write x^u for the corresponding character, considered as a function in $K[M]$.

4.2. Extended tropicalization. A point $\|\cdot\|$ in \mathbf{T}^{an} naturally determines a real valued linear function on the character lattice M , taking u to $-\log \|x^u\|$. The induced tropicalization map $\text{trop} : \mathbf{T}^{\text{an}} \rightarrow N_{\mathbf{R}}$ is continuous, proper, and surjective [Pay09a]. The image of $\mathbf{T}(K)$ is exactly N_G , which is dense in $N_{\mathbf{R}}$ because G is nontrivial and divisible.

More generally, if σ is a pointed rational polyhedral cone in $N_{\mathbf{R}}$ and $Y_{\sigma} = \text{Spec } K[\sigma^{\vee} \cap M]$ is the associated affine toric variety with dense torus \mathbf{T} , then there is a natural tropicalization map from Y_{σ} to the space of additive semigroup homomorphisms $\text{Hom}(\sigma^{\vee} \cap M, \mathbf{R} \cup \{\infty\})$ taking a point $\|\cdot\|$ to the semigroup map $u \mapsto -\log \|x^u\|$, where $-\log(0)$ is defined to be ∞ . See [Pay09a, Rab12] for further details. We write $N_{\mathbf{R}}(\sigma)$ for the image of Y_{σ}^{an} under this extended tropicalization map.

Definition 4.3. We say that a point in $N_{\mathbf{R}}(\sigma)$ is G -rational if it is in the subspace $\text{Hom}(\sigma^{\vee} \cap M, G \cup \{\infty\})$. Note that the image of any K -rational point of Y_{σ} is G -rational.

For any toric variety Y_{Δ} , the tropicalization $N_{\mathbf{R}}(\Delta)$ is the union of the spaces $N_{\mathbf{R}}(\sigma)$ for σ in Δ , glued along the open inclusions $N_{\mathbf{R}}(\tau) \subset N_{\mathbf{R}}(\sigma)$ for $\tau \preceq \sigma$. The tropicalization maps on torus invariant affine opens are compatible with this gluing, and together give a natural continuous, proper, and surjective map of topological spaces $\text{trop} : Y_{\Delta}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta)$. Note that the vector space $N_{\mathbf{R}}$, which is the tropicalization of the dense torus $\mathbf{T} \subset Y_{\Delta}$, is open and dense in $N_{\mathbf{R}}(\Delta)$. For the purpose of constructing tropical integral models of toric varieties and their subvarieties, it will generally suffice to study polyhedral complexes in $N_{\mathbf{R}}$.

Let X be a closed subscheme of Y_{Δ} . The tropicalization $\text{Trop}(X)$ is the image of X^{an} under trop . Since $X(K)$ is dense in X^{an} , its image is dense in $\text{Trop}(X)$. Furthermore, every G -rational point of $\text{Trop}(X)$ is the image of a point of $X(K)$, and if X is irreducible then the preimage of any point in $\text{Trop}(X) \cap N_G$ is Zariski dense in X . See [Pay09b, Corollary 4.2] and [Pay12, Remark 2], [Gub12, Proposition 4.14], or [OP10, Theorem 4.2.5].

4.4. Polyhedral domains. Recall that the *recession cone* σ_P of a nonempty polyhedron $P \subset N_{\mathbf{R}}$ is the set of those v in $N_{\mathbf{R}}$ such that $w + v$ is in P whenever w is in P . If P is the intersection of the halfspaces $\langle u_1, v \rangle \geq a_1, \dots, \langle u_r, v \rangle \geq a_r$ then σ_P is the dual of the cone in $M_{\mathbf{R}}$ spanned by u_1, \dots, u_r . In particular, if P is an integral G -affine polyhedron, then these halfspaces can be chosen with each u_i in M , so the recession cone σ_P is a rational polyhedral cone. The recession cone can also be characterized as the intersection with $N_{\mathbf{R}} \times \{0\}$ of the closure in $N_{\mathbf{R}} \times \mathbf{R}$ of the cone spanned by $P \times \{1\}$.

Let P be an integral G -affine polyhedron in $N_{\mathbf{R}}$ that does not contain any positive dimensional affine linear subspace, so its recession cone $\sigma = \sigma_P$ is pointed.

Definition 4.5. The *polyhedral domain* associated to P is the inverse image under $\text{trop} : Y_{\sigma}^{\text{an}} \rightarrow N_{\mathbf{R}}(\sigma)$ of the closure of P in $N_{\mathbf{R}}(\sigma)$ and is denoted \mathcal{U}^P .

These polyhedral domains, introduced in [Rab12], directly generalize the polytopal domains studied by Gubler in [Gub07b]. Indeed, a polytopal domain is the preimage in \mathbf{T}^{an} of an integral G -affine polytope in $N_{\mathbf{R}}$. Since the recession cone of a polytope in $N_{\mathbf{R}}$ is the zero cone, whose associated toric variety is \mathbf{T} , Gubler's polytopal domains are exactly the special case of these polyhedral domains where P is bounded.

By [Rab12, §6] the polyhedral domain \mathcal{U}^P is an affinoid domain in Y_{σ}^{an} with coordinate ring

$$K\langle \mathcal{U}^P \rangle = \left\{ \sum_{u \in \sigma^{\vee} \cap M} a_u x^u : \lim(\text{val}(a_u) + \langle u, v \rangle) = \infty \text{ for all } v \in P \right\},$$

where the limit is taken over all complements of finite sets. Its supremum norm is given by

$$(4.5.1) \quad \left| \sum a_u x^u \right|_{\sup} = \sup_{\substack{u \in \sigma^\vee \cap M \\ v \in P}} |a_u| \exp(-\langle u, v \rangle).$$

Since the recession cone σ is pointed, the polyhedron P contains no linear subspace and hence has vertices. The supremum above is always achieved at one of the vertices of P , so the ring of power-bounded regular functions on \mathcal{U}^P is

$$(4.5.2) \quad K\langle \mathcal{U}^P \rangle^\circ = \left\{ \sum_{u \in \sigma^\vee \cap M} a_u x^u \in K\langle \mathcal{U}^P \rangle : \text{val}(a_u) + \langle u, v \rangle \geq 0 \text{ for all } v \in \text{vert}(P) \right\}.$$

Since $K\langle \mathcal{U}^P \rangle$ is reduced, Theorem 3.17 implies that $\mathfrak{U}^P = \text{Spf}(K\langle \mathcal{U}^P \rangle^\circ)$ is an admissible formal scheme with analytic generic fiber \mathcal{U}^P .

Remark 4.6. If P is integral affine but not G -affine then the inverse image \mathcal{U}^P of the closure of P under trop is a non-strict affinoid domain. Indeed, if K' is a complete valued field extension of K whose value group G' is large enough that P is G' -affine then $\mathcal{U}^P \widehat{\otimes}_K K'$ is strictly K' -affinoid.

4.7. Polyhedral integral models. Let P be an integral G -affine polyhedron in $N_{\mathbf{R}}$ whose recession cone $\sigma = \sigma_P$ is pointed. As usual, we let $Y_\sigma = \text{Spec } K[\sigma^\vee \cap M]$ denote the associated affine toric variety with dense torus \mathbf{T} .

Definition 4.8. We define $R[Y^P] \subset K[\sigma^\vee \cap M]$ to be the subring consisting of those Laurent polynomials $\sum a_u x^u$ such that $\text{val}(a_u) + \langle u, v \rangle \geq 0$ for all $v \in P$ and all u . The scheme $Y^P := \text{Spec}(R[Y^P])$ is called a *polyhedral integral model* of Y_σ .

In other words, $R[Y^P]$ is the intersection of $K\langle \mathcal{U}^P \rangle^\circ$ with $K[M]$. It is clear that $K\langle \mathcal{U}^P \rangle^\circ$ is the ϖ -adic completion of $R[Y^P]$. Note that $R[Y^P]$ is torsion-free and hence flat over R .

Lemma 4.9. *The tensor product $R[Y^P] \otimes_R K$ is equal to $K[Y_\sigma]$.*

Proof. By definition we have $R[Y^P] \otimes_R K \subset K[Y_\sigma]$. For the other inclusion, note that if $g = \sum b_u x^u$ is in $K[Y_\sigma]$ then the minimum over v in P of $\text{val}(b_u) + \langle u, v \rangle$ is achieved at some vertex of P . It follows that some sufficiently high power of ϖ times g is in $R[Y^P]$, and hence g is in $R[Y^P] \otimes_R K$. \blacksquare

Remark 4.10. One could equivalently define $R[Y^P]$ to be the subring of $K[M]$ satisfying the same inequalities. Since P is closed under addition of points in σ , any Laurent polynomial satisfying these inequalities for all v in P must be supported in σ^\vee .

We will use the following notation in the proof of Proposition 4.11 below. For each face $F \leq P$, let $\sigma(F)$ be the cone in $N_{\mathbf{R}}$ spanned by $P - v$ for any v in the relative interior of F . In other words, $\sigma(F) = \text{Star}_P(F)$. We fix a labeling v_1, \dots, v_r for the vertices of P , and write σ_i for $\sigma(v_i)$. The dual cone σ_i^\vee is

$$\sigma_i^\vee = \{u \in \sigma_P^\vee : \langle u, v_i \rangle \leq \langle u, v_j \rangle \text{ for all } j\}.$$

The cones $\sigma_1^\vee, \dots, \sigma_r^\vee$ are the maximal cones of the (possibly degenerate) inner normal fan of P , and their union is σ_P^\vee .

Proposition 4.11. *Let P be a G -rational polyhedron in $N_{\mathbf{R}}$. Then $R[Y^P]$ is finitely presented over R .*

Proof. By [RG71, Corollary 3.4.7], any finitely generated and flat algebra over an integral domain is automatically of finite presentation, so it suffices to show that $R[Y^P]$ is finitely generated.

The cones $\sigma_1^\vee, \dots, \sigma_r^\vee$ cover σ_P^\vee , so $R[Y^P]$ is generated by the subrings

$$A_j = R[Y^P] \cap K[\sigma_j^\vee \cap M]$$

for $1 \leq j \leq r$. Therefore, it will suffice to show that each A_j is finitely generated over R .

The semigroup $\sigma_j^\vee \cap M$ is finitely generated by Gordan's Lemma [Ful93, p. 12]. Let u_1, \dots, u_s be generators, and choose a_1, \dots, a_s in K^\times such that $\text{val}(a_i) + \langle u_i, v_j \rangle = 0$. Then each monomial in

A_j can be written as an element of R times a monomial in the $a_i x^{u_i}$. It follows that A_j is finitely generated over R , as required, with generating set $\{a_1 x^{u_1}, \dots, a_s x^{u_s}\}$. ■

In particular, Y^P is a flat and finitely presented R -model of the affine toric variety Y_σ .

Remark 4.12. As in Remark 4.6, one can construct an algebraic model Y^P of Y_σ associated to an integral affine but not G -affine polyhedron P ; when P is a point this is done in [OP10]. This model is not of finite type.

4.13. Polyhedral integral and formal models of subschemes. Let P be an integral G -affine polyhedron with pointed recession cone σ . Let X be the closed subscheme of the affine toric variety Y_σ over K defined by an ideal $\mathfrak{a} \subset K[Y_\sigma]$.

Definition 4.14.

- (1) Let $\mathcal{X}^P = X^{\text{an}} \cap \mathcal{U}^P$. This is the Zariski-closed subspace of \mathcal{U}^P defined by $\mathfrak{a}K\langle\mathcal{U}^P\rangle$.
- (2) The *polyhedral integral model* of X is the scheme-theoretic closure X^P of X in Y^P . It is defined by the ideal $\mathfrak{a}^P = \mathfrak{a} \cap R[Y^P]$.
- (3) The *polyhedral formal model* of \mathcal{X}^P is the ϖ -adic completion \mathfrak{X}^P of X^P . We will show in Proposition 4.17 that \mathfrak{X}^P is an admissible formal scheme with generic fiber \mathcal{X}^P .
- (4) The *canonical model* of \mathcal{X}^P is

$$\mathfrak{X}_{\text{can}}^P = \text{Spf}((K\langle\mathcal{U}^P\rangle/\mathfrak{a}K\langle\mathcal{U}^P\rangle)^\circ).$$

By Theorem 3.17, the canonical model is admissible if and only if \mathcal{X}^P is reduced.

Notation 4.15. The P -initial degeneration of X is defined to be

$$\text{in}_P(X) = X^P \otimes_R k = \mathfrak{X}^P \otimes_R k.$$

As usual we write $\overline{\mathfrak{X}}_{\text{can}}^P = \mathfrak{X}_{\text{can}}^P \otimes_R k$. This coincides with the canonical reduction of \mathcal{X}^P when \mathcal{X}^P is reduced. In the case where P is a single point $w \in N_G$ we write \mathcal{X}^w , \mathfrak{X}^w , $\text{in}_w(X)$, etc. In this case, $\text{in}_w(X)$ is the w -initial degeneration of X in the sense generally used in the literature (and in the introduction).

Lemma 4.16. *The ideal \mathfrak{a}^P is finitely generated.*

Proof. Since X^P is the closure of its generic fiber, it is flat over $\text{Spec } R$, and its coordinate ring is a quotient of the finitely generated R -algebra $R[M]$. Since any finitely generated flat algebra over an integral domain is finitely presented [RG71, Corollary 3.4.7], it follows that \mathfrak{a}^P is finitely generated. ■

Proposition 4.17. *The formal scheme \mathfrak{X}^P is the formal closed subscheme of \mathfrak{U}^P defined by $\mathfrak{a}^P K\langle\mathcal{U}^P\rangle^\circ$. It is an admissible formal scheme with generic fiber \mathcal{X}^P and special fiber $\text{in}_P(X)$.*

Proof. The admissibility of \mathfrak{X}^P is a consequence of Proposition 3.12(1). If $A = R[Y^P]/\mathfrak{a}$ then by definition $X^P = \text{Spec}(A)$ and $\mathfrak{X}^P = \text{Spf}(\widehat{A})$, where \widehat{A} is the ϖ -adic completion of A . By Proposition 3.12(2) the sequence

$$0 \longrightarrow \mathfrak{a}^P K\langle\mathcal{U}^P\rangle^\circ \longrightarrow K\langle\mathcal{U}^P\rangle^\circ \longrightarrow \widehat{A} \longrightarrow 0$$

is exact; it follows that \mathfrak{X}^P is the closed subscheme of \mathfrak{U}^P defined by $\mathfrak{a}^P K\langle\mathcal{U}^P\rangle^\circ$. We have

$$(K\langle\mathcal{U}^P\rangle^\circ/\mathfrak{a}^P K\langle\mathcal{U}^P\rangle^\circ) \otimes_R K = K\langle\mathcal{U}^P\rangle/\mathfrak{a}K\langle\mathcal{U}^P\rangle$$

since $K\mathfrak{a}^P = \mathfrak{a}$, so $\mathfrak{X}^P \otimes_R K = \mathcal{X}^P$. The special fiber of \mathfrak{X}^P agrees with the special fiber of X^P by construction. ■

The canonical inclusion

$$K\langle\mathcal{U}^P\rangle^\circ/\mathfrak{a}^P K\langle\mathcal{U}^P\rangle^\circ \hookrightarrow (K\langle\mathcal{U}^P\rangle/\mathfrak{a}K\langle\mathcal{U}^P\rangle)^\circ$$

induces a map of formal schemes

$$\mathfrak{X}_{\text{can}}^P \longrightarrow \mathfrak{X}^P.$$

As the above morphism induces an isomorphism on analytic generic fibers, it is *finite* when \mathcal{X}^P is reduced by Proposition 3.13(2). The special fiber of the above morphism is a morphism $\overline{\mathfrak{X}}_{\text{can}}^P \rightarrow \text{in}_P(X)$. Many of the results of this paper are proved by using this morphism and the results of §3 (in particular the projection formula, Proposition 3.32) to compare these two models.

4.18. Compatibility with extension of the ground field. We continue to use the notation of (4.13). Let K' be an algebraically closed complete valued field extension of K , with valuation ring R' and residue field k' . Let P be an integral G -affine polyhedron in $N_{\mathbf{R}}$ with pointed recession cone σ . Let $Y'_\sigma = Y_\sigma \otimes_K K'$, so Y'_σ is the affine toric variety defined over K' with dense torus $\mathbf{T}' := \mathbf{T} \otimes_K K'$ associated to the cone σ . The triangle

$$\begin{array}{ccc} (Y'_\sigma)^{\text{an}} & \xrightarrow{\quad} & Y_\sigma^{\text{an}} \\ & \searrow \text{trop} \quad \swarrow \text{trop} & \\ & N_{\mathbf{R}}(\sigma) & \end{array}$$

commutes, so $\mathcal{U}^P \widehat{\otimes}_K K'$ is the polyhedral domain in $(Y'_\sigma)^{\text{an}}$ associated to P . Likewise the polyhedral integral model $(Y'_\sigma)^P$ of Y'_σ associated to P is naturally identified with $Y^P \otimes_R R'$. Indeed, as an R -module we have

$$R[Y^P] = \bigoplus_{u \in \sigma^\vee \cap M} R_u \cdot x^u \subset K[\sigma^\vee \cap M] \quad \text{where} \quad R_u = \left\{ a \in R : \text{val}(a) \geq \max_{v \in \text{vert}(P)} -\langle u, v \rangle \right\}.$$

Since $\langle u, v \rangle \in G$ for all $u \in M$ and $v \in \text{vert}(P)$ each R_u is a free R -module of rank 1, so the image of $R_u \otimes_R R'$ in K' is exactly R'_u .

Let $X \subset Y_\sigma$ be the closed subscheme defined by an ideal $\mathfrak{a} \subset K[\sigma^\vee \cap M]$ and let $X' = X \otimes_K K' \subset Y'_\sigma$, so X' is defined by $\mathfrak{a}K'[\sigma^\vee \cap M]$. Since the above triangle is commutative, we have $\text{Trop}(X) = \text{Trop}(X') \subset N_{\mathbf{R}}(\sigma)$, and $\text{trop} : (X')^{\text{an}} \rightarrow \text{Trop}(X)$ factors through the natural map $(X')^{\text{an}} \rightarrow X^{\text{an}}$. Hence

$$(\mathcal{X}')^P = \text{trop}^{-1}(P) \cap (X')^{\text{an}} = \mathcal{X}^P \widehat{\otimes}_K K'.$$

Since schematic closure commutes with flat base change, the polyhedral integral model $(X')^P$ of X' coincides with $X^P \otimes_R R'$; hence if $\mathfrak{a}^P = \mathfrak{a} \cap R[Y^P]$ is the ideal defining X^P then $(X')^P$ is defined by $\mathfrak{a}^P R'[(Y')^P]$. It follows from this and Proposition 4.17 that $(\mathfrak{X}')^P = \mathfrak{X}^P \widehat{\otimes}_R R'$, and in particular that $\text{in}_P(X') = \text{in}_P(X) \otimes_k k'$. As for the canonical models, suppose that X is reduced, so X' is reduced as well. Then $(\mathfrak{X}')_{\text{can}}^P = \mathfrak{X}_{\text{can}}^P \widehat{\otimes}_R R'$ because $(\mathfrak{X}_{\text{can}}^P \widehat{\otimes}_R R') \otimes_{R'} k' = (\mathfrak{X}_{\text{can}}^P \otimes_R k) \otimes_k k'$ is reduced; cf. Proposition 3.18.

Below we will make various definitions by passing to a valued field extension K' of K . In order for these definitions to be independent of the choice of K' , we will need the following fact, proven in [Duc09, 0.3.2] or [Con08, §4].

Lemma 4.19. *Let K_1, K_2 be complete valued field extensions of K . Then there is a complete valued field extension K' of K admitting isometric embeddings $K_1 \hookrightarrow K'$ and $K_2 \hookrightarrow K'$ over K .*

4.20. Relative multiplicities and tropical multiplicities. Recall (3.8) that if $\mathcal{X} = \mathcal{M}(\mathcal{A})$ is an affinoid space then the reduction map induces a one-to-one correspondence between the Shilov boundary points of \mathcal{X}^{an} and the generic points of the canonical reduction $\text{Spec}(\tilde{\mathcal{A}})$. This leads to the following definition:

Definition 4.21. Let $X \subset \mathbf{T}$ be a reduced and equidimensional closed subscheme, let $x \in X^{\text{an}}$, let $w = \text{trop}(x)$, and suppose that $w \in N_G$. Define the *relative multiplicity* $m_{\text{rel}}(x)$ of x in $\text{trop}^{-1}(w)$ as follows. If x is not a Shilov boundary point of $\text{trop}^{-1}(w)$ then we define its multiplicity to be zero. Otherwise $\text{red}(x)$ is the generic point of an irreducible component \mathfrak{C} of $\mathfrak{X}_{\text{can}}^w$; we define the multiplicity of x to be $[\mathfrak{C} : \text{im}(\mathfrak{C})]$, where $\text{im}(\mathfrak{C})$ is the image of \mathfrak{C} in $\text{in}_w(X)$ (this is an irreducible component by Corollary 3.16).

Now suppose that $w \notin N_G$. Let K' be an algebraically closed complete valued field extension of K such that $w \in N_{G'}$, where G' is the value group of K' . Let $X' = X \otimes_K K'$ and let $\varphi : (X')^{\text{an}} \rightarrow X^{\text{an}}$

be the natural morphism. We define

$$m_{\text{rel}}(x) = \sum_{x' \in \varphi^{-1}(x)} m_{\text{rel}}(x').$$

In order for the above definition to make sense, by Lemma 4.19 we only have to show that if $K \subset K' \subset K''$ are algebraically closed complete valued field extensions then we can calculate $m_{\text{rel}}(x)$ with respect to either K' or K'' . Replacing K with K' , we are reduced to showing:

Lemma 4.22. *Let K' be an algebraically closed complete valued field extension of K and let $X' = X \otimes_K K'$. Let $x \in X^{\text{an}}$, let $w = \text{trop}(x)$, and suppose that $w \in N_G$. Then the natural map $(\mathcal{X}')^w \rightarrow \mathcal{X}^w$ induces a bijection of Shilov boundary points which preserves relative multiplicities.*

Proof. Let k' be the residue field of K' . As discussed in (4.18) we have $\text{in}_w(X') = \text{in}_w(X) \otimes_k k'$ and $(\overline{\mathfrak{X}}')_{\text{can}}^w = \overline{\mathfrak{X}}_{\text{can}}^w \otimes_k k'$, so the first assertion follows from the fact that $\overline{\mathfrak{X}}_{\text{can}}^w \otimes_k k' \rightarrow \overline{\mathfrak{X}}_{\text{can}}^w$ induces a bijection on irreducible components. Let $\overline{\mathfrak{C}}$ be an irreducible component of $\overline{\mathfrak{X}}_{\text{can}}^w$ and let $\overline{\mathfrak{D}}$ be its image in $\overline{\mathfrak{X}}^w$. Then $[\overline{\mathfrak{C}} : \overline{\mathfrak{D}}] = [\overline{\mathfrak{C}} \otimes_k k' : \overline{\mathfrak{D}} \otimes_k k']$, so relative multiplicities are preserved as well. \blacksquare

Later we will relate $m_{\text{rel}}(x)$ to other geometrically-defined notions of multiplicity; see Proposition 4.32 and Theorem 6.8. For the moment we relate relative multiplicities to tropical multiplicities, defined as follows:

Definition 4.23. Let $X \subset \mathbf{T}$ be a closed subscheme and let $w \in \text{Trop}(X)$. If $w \in N_G$ then the *tropical multiplicity of X at w* is defined to be

$$m_{\text{Trop}}(w) = \sum_{\overline{C} \subset \text{in}_w(X)} \text{mult}_{\text{in}_w(X)}(\overline{C}),$$

where the sum is taken over all irreducible components \overline{C} of $\text{in}_w(X)$. If $w \notin N_G$ then let K' be an algebraically closed complete valued field extension of K such that $w \in N_{G'}$, where G' is the value group of K' . Let $X' = X \otimes_K K'$. We define $m_{\text{Trop}}(w)$ to be the tropical multiplicity of w relative to $\text{trop} : (X')^{\text{an}} \rightarrow \text{Trop}(X)$.

The fact that $m_{\text{Trop}}(w)$ is independent of the choice of K' is proved in [OP10, Remark A.5]. It is also one of the consequences of the following proposition.

Proposition 4.24. *Let $X \subset \mathbf{T}$ be a reduced and equidimensional closed subscheme and let $w \in \text{Trop}(X)$. Then*

$$m_{\text{Trop}}(w) = \sum_{x \in \text{trop}^{-1}(w)} m_{\text{rel}}(x).$$

Proof. We immediately reduce to the case where $w \in N_G$ by extending the ground field if necessary. By definition we have

$$\sum_{x \in \text{trop}^{-1}(w)} m_{\text{rel}}(x) = \sum_{\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}_{\text{can}}^w} [\overline{\mathfrak{C}} : \text{im}(\overline{\mathfrak{C}})]$$

where the sum is taken over all irreducible components $\overline{\mathfrak{C}}$ of $\overline{\mathfrak{X}}_{\text{can}}^w$; the image $\text{im}(\overline{\mathfrak{C}})$ of $\overline{\mathfrak{C}}$ in $\text{in}_w(X)$ is an irreducible component by Corollary 3.16. Also by definition,

$$m_{\text{Trop}}(w) = \sum_{\overline{C} \subset \text{in}_w(X)} \text{mult}_{\text{in}_w(X)}(\overline{C})$$

where the sum is taken over all irreducible components \overline{C} of $\text{in}_w(X)$. By the projection formula (3.32), for every irreducible component \overline{C} of $\text{in}_w(X)$ we have

$$\text{mult}_{\text{in}_w(X)}(\overline{C}) = \sum_{\overline{\mathfrak{C}} \rightarrow \overline{C}} [\overline{\mathfrak{C}} : \overline{C}]$$

where the sum is taken over all irreducible components $\bar{\mathcal{C}}$ of $\bar{\mathfrak{X}}_{\text{can}}^w$ mapping onto \bar{C} (for any such $\bar{\mathcal{C}}$ we have $\text{mult}_{\bar{\mathfrak{X}}_{\text{can}}}(\bar{\mathcal{C}}) = 1$ since $\bar{\mathfrak{X}}_{\text{can}}^w$ is reduced). Therefore

$$m_{\text{Trop}}(w) = \sum_{\bar{C} \subset \text{in}_w(X)} \text{mult}_{\text{in}_w(X)}(\bar{C}) = \sum_{\bar{C} \subset \text{in}_w(X)} \sum_{\bar{\mathcal{C}} \rightarrow \bar{C}} [\bar{\mathcal{C}} : \bar{C}] = \sum_{\bar{\mathcal{C}} \subset \bar{\mathfrak{X}}_{\text{can}}^w} [\bar{\mathcal{C}} : \text{im}(\bar{\mathcal{C}})].$$

■

Example 4.25. The following example illustrates the above definitions and Proposition 4.24. Consider the curve X in the 2-dimensional torus \mathbf{G}_m^2 over $K = \mathbf{C}_p$ given by $y^2 + (2-p)y = x + (p-1)$ (cf. Example 2.6). We determine \mathfrak{X}^w and $\bar{\mathfrak{X}}_{\text{can}}^w$ explicitly for $w = (\beta, 0) \in \text{Trop}(X) \cap N_{\mathbf{Q}}$ when $\beta \geq 0$. First of all, by a simple change of variables in x we have

$$\mathcal{X}^w = \text{trop}^{-1}(w) \cong \mathcal{M}(K\langle x, y, x^{-1}, y^{-1} \rangle / (y^2 + (2-p)y = p^\beta x + (p-1))).$$

The equation $y^2 + (2-p)y = p^\beta x + (p-1)$ can be rewritten as $(y - (p-1))(y+1) = p^\beta x$, which implies that $\text{val}(y - (p-1)) + \text{val}(y+1) = \beta$ for any $(x, y) \in \mathcal{X}^w$. For $0 \leq \beta < 2$, the ultrametric inequality shows that \mathcal{X}^w is isomorphic to the analytic zero-modulus annulus whose K -points are

$$\{y \in K : \text{val}(y+1) = \beta/2\} = \{y \in K : \text{val}(y - (p-1)) = \beta/2\}.$$

More formally, there is an isomorphism of K -analytic spaces $\mathcal{X}^w \cong \mathcal{M}(K\langle T, T^{-1} \rangle)$ corresponding to the isomorphism of K -affinoid algebras

$$K\langle x, y, x^{-1}, y^{-1} \rangle / (y^2 + (2-p)y = p^\beta x + (p-1)) \rightarrow K\langle T, T^{-1} \rangle$$

given by $x \mapsto T^2 - p^{\beta/2}T$ and $y \mapsto p^{\beta/2}T - 1$. The inverse of this map is $T \mapsto p^{-\beta/2}(y+1)$.

We have

$$\mathfrak{X}_{\text{can}}^w \cong \text{Spf } R\langle T, T^{-1} \rangle$$

and

$$\mathfrak{X}^w \cong \text{Spf}(R\langle x, y, x^{-1}, y^{-1} \rangle / (y^2 + (2-p)y = p^\beta x + (p-1)))$$

with the canonical map $\bar{\mathfrak{X}}_{\text{can}}^w \rightarrow \mathfrak{X}^w$ given by $x \mapsto T^2 - p^{\beta/2}T$ and $y \mapsto p^{\beta/2}T - 1$. For $w = (0, 0)$, the induced map $\bar{\mathfrak{X}}_{\text{can}}^w \rightarrow \text{in}_w(X)$ on special fibers is an isomorphism (both sides are isomorphic to $\mathbf{G}_{m,k}$). In this case $m_{\text{Trop}}(w) = m_{\text{rel}}(\xi) = 1$ where ξ is the unique Shilov boundary point of \mathcal{X}^w . For $w = (\beta, 0)$ with $0 < \beta < 2$, the initial degeneration $\text{in}_w(X) \cong \text{Spec}(k[x, x^{-1}][y+1]/(y+1)^2)$ is a length-2 nilpotent thickening of $\mathbf{G}_{m,k}$, and $\bar{\mathfrak{X}}_{\text{can}}^w \cong \text{Spec}(k[T, T^{-1}]) \cong \mathbf{G}_m$, with the map being $x \mapsto T^2$. In this case $m_{\text{Trop}}(w) = m_{\text{rel}}(\xi) = 2$, with ξ as above.

For $\beta \geq 2$, the initial degeneration $\text{in}_w(X)$ is the same as in the case $0 < \beta < 2$, namely $\text{in}_w(X) \cong \text{Spec}(k[x, x^{-1}][y+1]/(y+1)^2)$, so $m_{\text{Trop}}(w) = 2$. However, when $\beta > 2$, the ultrametric inequality shows that \mathcal{X}^w is isomorphic to a disjoint union of two zero-modulus annuli:

$$\mathcal{X}^w(K) = \{y \in K : \text{val}(y+1) = 1\} \amalg \{y \in K : \text{val}(y+1) = \beta - 1\}.$$

In this case, $\bar{\mathfrak{X}}_{\text{can}}^w$ is isomorphic to a disjoint union of two copies of $\mathbf{G}_{m,k}$ and the natural map $\bar{\mathfrak{X}}_{\text{can}}^w \rightarrow \text{in}_w(X)$ is an isomorphism from each connected component of $\bar{\mathfrak{X}}_{\text{can}}^w$ onto its image. In particular there are two Shilov boundary points of \mathcal{X}^w , each with relative multiplicity one.

For $\beta = 2$, similar computations show that \mathcal{X}^w is isomorphic to the closed ball of radius $1/p$ around -1 with the open balls of radius $1/p$ around -1 and $p-1$ removed. In this case, $\bar{\mathfrak{X}}_{\text{can}}^w$ is irreducible and the natural map $\bar{\mathfrak{X}}_{\text{can}}^w \rightarrow \text{in}_w(X)$ has degree 2 onto its image.

Example 4.26. In Example 2.8, we have $m_{\text{rel}}(\xi) = 1$ for all $\xi \in \Gamma$ and $m_{\text{rel}}(\xi) = 0$ for all $\xi \notin \Gamma$. This follows from Proposition 4.24 and the concrete description of the tropicalization map in Example 2.8, together with the observation that $m_{\text{Trop}}(0, 0) = m_{\text{Trop}}(2, 2) = 1$ (since the initial degenerations $\text{in}_{(0,0)}(E) \cong \text{Spec } k[x, y, x^{-1}, y^{-1}]/(y^2 - x^3 - x^2)$ and $\text{in}_{(2,2)}(E) \cong \text{Spec } k[x, y, x^{-1}, y^{-1}]/(y^2 - x^2 - 1)$ are both integral schemes over k). Note that $m_{\text{rel}}(\xi) > 0$ for all $\xi \in \Gamma$ because ξ is contained in the topological boundary of $\text{trop}^{-1}(\text{trop}(\xi))$ in E^{an} , hence in the Shilov boundary; see (3.9).

4.27. Polyhedral structures on tropicalizations. Let W be a G -rational affine space in $N_{\mathbf{R}}$ and let W_0 be the linear space under W , so W_0 is spanned by $W_0 \cap N$. Set $N' = N/(W_0 \cap N)$ and $M' = W_0^\perp \cap M \subset M$, and let \mathbf{T}' be the torus $\text{Spec}(K[M'])$. We call \mathbf{T}' the *torus transverse to W* . Let $w' \in N'_G$ be the image of any point of W . Then

$$R[(\mathbf{T}')^{w'}] = \left\{ \sum_{u \in M'} a_u x^u \in K[M'] : \text{val}(a_u) + \langle u, w' \rangle \geq 0 \right\},$$

so for all $w \in N_G \cap W$ we have $R[(\mathbf{T}')^{w'}] \subset R[\mathbf{T}^w]$. Hence we have a natural morphism $\pi_w : \overline{\mathbf{T}}^w \rightarrow (\overline{\mathbf{T}}')^{w'}$ for all $w \in N_G \cap W$.

Remark 4.28. Let $N'' = \ker(N \rightarrow N') = W_0 \cap N$ and let $M'' = \text{Hom}_{\mathbf{Z}}(N'', \mathbf{Z})$, so we have exact sequences

$$0 \rightarrow N'' \rightarrow N \rightarrow N' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

inducing an exact sequence of tori

$$0 \rightarrow \mathbf{T}'' \rightarrow \mathbf{T} \rightarrow \mathbf{T}' \rightarrow 0$$

where $\mathbf{T}'' = \text{Spec}(K[M''])$. We call \mathbf{T}'' the *torus parallel to W* . Choosing a splitting of $N \rightarrow N'$ splits all three exact sequences, and in particular furnishes an isomorphism $\mathbf{T} \cong \mathbf{T}' \times \mathbf{T}''$. Let $w \in W \cap N_G$ and let w'' be its image in N''_G . Then we have an isomorphism $\overline{\mathbf{T}}^w \cong (\overline{\mathbf{T}}')^{w'} \times (\overline{\mathbf{T}}'')^{w''}$ under which π_w corresponds to the projection onto the first factor.

Theorem 4.29. *Let $X \subset \mathbf{T}$ be an equidimensional subscheme of dimension d . The set $\text{Trop}(X)$ admits a polyhedral complex structure of pure dimension d with the following properties:*

- (1) *The tropical multiplicities are constant along the relative interior of every maximal face.*
- (2) *Let w be contained in the relative interior of a maximal face τ of $\text{Trop}(X)$, let $W = \text{span}(\tau)$, let \mathbf{T}' be the torus transverse to W , and let $\pi_w : \overline{\mathbf{T}}^w \rightarrow (\overline{\mathbf{T}}')^{w'}$ be the natural map. Then $\text{in}_w(X) \cong \pi_w^{-1}(\overline{Y})$ for some dimension-zero subscheme \overline{Y} of $(\overline{\mathbf{T}}')^{w'}$.*

Proof. The first part is a basic result in tropical geometry; it is proved in [MS09, §3.3].⁶ Let $\mathbf{T}'' \subset \mathbf{T}$ be the torus parallel to W and write $\overline{\mathbf{T}}'' = (\overline{\mathbf{T}}'')^0$, so $(\overline{\mathbf{T}}')^{w'}$ is the quotient of $\overline{\mathbf{T}}^w$ by $\overline{\mathbf{T}}''$. By [Spe05, Proposition 2.2.4], the initial degeneration $\text{in}_w(X)$ is invariant under the action of $\overline{\mathbf{T}}''$. Therefore $\text{in}_w(X)$ is the inverse image of a closed subscheme \overline{Y} of $(\overline{\mathbf{T}}')^{w'}$ (in fact \overline{Y} is the quotient $\text{in}_w(X)/\overline{\mathbf{T}}'' \subset (\overline{\mathbf{T}}')^{w'}$); counting dimensions, we see that $\dim(\overline{Y}) = 0$. See [MFK94, Theorem 1.1.1 and Amplification 1.1.3] for basic existence results about geometric quotients of affine schemes over a field by a free action of a reductive group. \blacksquare

In the situation of Theorem 4.29(2), let \mathbf{T}'' be the torus parallel to W , and choose a splitting $\mathbf{T} \cong \mathbf{T}' \times \mathbf{T}''$ as in Remark 4.28. Then $\text{in}_w(X) \cong \overline{Y} \times (\overline{\mathbf{T}}'')^{w''}$.

4.30. The tropical projection formula. Let $X \subset \mathbf{T}$ be a reduced and equidimensional closed subscheme of dimension d and let P be an integral G -affine polytope contained in the relative interior of a maximal (d -dimensional) face τ of a polyhedral complex decomposition of $\text{Trop}(X)$ as in Theorem 4.29. Let W be the affine span of τ , let \mathbf{T}' be the torus transverse to W (4.27), let \mathbf{T}'' be the torus parallel to W (Remark 4.28), and choose a splitting $\mathbf{T} \rightarrow \mathbf{T}' \times \mathbf{T}''$. Note that $\dim(\mathbf{T}'') = d$. Let P'' be the image of P in $N''_{\mathbf{R}}$, so $\mathcal{U}^{P''}$ is a polytopal domain in $(\mathbf{T}'')^{\text{an}}$. The map $\mathcal{U}^P \rightarrow \mathcal{U}^{P''}$ induces a morphism $\psi^P : \mathcal{X}^P \rightarrow \mathcal{U}^{P''}$.

Theorem 4.31. *The morphism $\psi^P : \mathcal{X}^P \rightarrow \mathcal{U}^{P''}$ is finite, and every irreducible component of \mathcal{X}^P surjects onto $\mathcal{U}^{P''}$.*

⁶The proofs in [MS09, §3.3] assume that there is a section to the valuation map $\text{val} : K^\times \rightarrow G$. Such a section always exists when K is an algebraically closed nonarchimedean field; the following short proof was communicated to us by David Speyer. If $G = \{0\}$ then there is nothing to prove. Otherwise, consider the short exact sequence $0 \rightarrow U \rightarrow K^\times \rightarrow G \rightarrow 0$. Since K is algebraically closed, the group U is divisible. Thus U is injective as a \mathbf{Z} -module, so $\text{Ext}^1(A, U) = 0$ and the valuation map splits.

Proof. For dimension reasons it suffices to show that ψ^P is finite. Since $\mathcal{X}^P = \mathcal{M}(\mathcal{A})$ and $\mathcal{U}^P = \mathcal{M}(K\langle \mathcal{U}^{P''} \rangle)$ are both affinoid, by the rigid-analytic direct image theorem [BGR84, Theorem 9.6.3/1] it suffices to show that $\mathcal{X}^P \rightarrow \mathcal{U}^{P''}$ is proper in the sense of [BGR84, §9.6.2]. In fact we will show that $\mathcal{X}^P \subseteq_{\mathcal{U}^{P''}} \mathcal{X}^P$, i.e., that there exist affinoid generators f_1, \dots, f_r for \mathcal{A} over $K\langle \mathcal{U}^{P''} \rangle$ such that $|f_1|_{\sup}, \dots, |f_r|_{\sup} < 1$.

Choosing bases for N' and N'' , we obtain isomorphisms $N'_{\mathbf{R}} \cong \mathbf{R}^{n'}$, $N''_{\mathbf{R}} \cong \mathbf{R}^d$, and $N_{\mathbf{R}} \cong \mathbf{R}^{n'} \times \mathbf{R}^d$. Translating by an element of $\mathbf{T}(K)$, we may and do assume that $P \subset \{0\} \times N''_{\mathbf{R}}$ (so $P = P''$). For $\varepsilon \in G$ with $\varepsilon > 0$ we let $I_\varepsilon \subset N'_{\mathbf{R}}$ be the cube $[-\varepsilon, \varepsilon]^{n'}$, so I_ε is a integral G -affine polytope in $N'_{\mathbf{R}}$, and $P_\varepsilon := I_\varepsilon \times P''$ is a integral G -affine polytope in $N_{\mathbf{R}} = N'_{\mathbf{R}} \times N''_{\mathbf{R}}$ containing P . Since τ is a maximal face we have $P_\varepsilon \cap \text{Trop}(X) = P$ for small ε ; we fix such an ε as well as an element $e \in K$ with $\text{val}(e) = \varepsilon$. The polytopal subdomain $\mathcal{U}^{I_\varepsilon} \subset (\mathbf{T}')^{\text{an}}$ is a product of annuli of inner radius $|e|$ and outer radius $|e|^{-1}$, so if $u_1, \dots, u_{n'}$ is a basis for M' then $\{ex^{\pm u_1}, \dots, ex^{\pm u_{n'}}\}$ is a set of affinoid generators for $K\langle \mathcal{U}^{I_\varepsilon} \rangle$. Since $\mathcal{U}^{P_\varepsilon} = \mathcal{U}^{I_\varepsilon} \times_K \mathcal{U}^{P''}$, it follows that $\{ex^{\pm u_1}, \dots, ex^{\pm u_{n'}}\}$ is a set of affinoid generators for $K\langle \mathcal{U}^{P_\varepsilon} \rangle$ over $K\langle \mathcal{U}^{P''} \rangle$. Since $P_\varepsilon \cap \text{Trop}(X) = P$ we have $X^{\text{an}} \cap \mathcal{U}^{P_\varepsilon} = \mathcal{X}^P$, so $\{ex^{\pm u_1}, \dots, ex^{\pm u_{n'}}\}$ can be regarded as a set of affinoid generators for \mathcal{A} over $K\langle \mathcal{U}^{P''} \rangle$. But by construction $|x^{u_i}(x)| = 1$ for all $x \in \mathcal{X}^P$ and all $i = 1, \dots, n'$, so $|ex^{u_i}(x)| = |e| < 1$. This proves that ψ^P is finite. \blacksquare

It follows from Theorem 4.31 and Remarks 3.24(1) and 3.29(1) that ψ^P has a (pure) degree.

Proposition 4.32. *In the situation of (4.30), let $\mathcal{Y} \subset \mathcal{X}^P$ be a union of connected components and let $w \in P$. Then*

$$[\mathcal{Y} : \mathcal{U}^{P''}] = \sum_{x \in \mathcal{Y} \cap \text{trop}^{-1}(w)} m_{\text{rel}}(x).$$

Proof. Extending the ground field if necessary, we assume that $w \in N_G$. Let w'' be the image of w in N''_G . Since $\mathcal{Y} \cap \mathcal{X}^w \rightarrow \mathcal{U}^{w''}$ is obtained by flat base change from $\mathcal{Y} \rightarrow \mathcal{U}^{P''}$ we may replace P by w and P'' by w'' to assume that $\mathcal{Y} \subset \mathcal{X}^w$ (cf. Proposition 3.30(1)). Let \mathfrak{Y} be the canonical model of Y . The canonical reduction $\overline{\mathfrak{Y}}$ of \mathfrak{Y} is a union of connected components of $\overline{\mathfrak{X}}_{\text{can}}^w$, so for $x \in \mathcal{Y}$ the relative multiplicity $m_{\text{rel}}(x)$ is nonzero if and only if $\text{red}(x)$ is the generic point of an irreducible component $\overline{\mathfrak{C}}$ of $\overline{\mathfrak{Y}}$, in which case $m_{\text{rel}}(x) = [\overline{\mathfrak{C}} : \text{im}(\overline{\mathfrak{C}})]$ where $\text{im}(\overline{\mathfrak{C}})$ is the image of $\overline{\mathfrak{C}}$ in $\text{in}_w(X)$. Noting that $\overline{\mathfrak{U}}^{w''}$ is an integral domain and $\overline{\mathfrak{Y}}$ is reduced, applying the projection formula (3.32) to $\mathfrak{Y} \rightarrow \mathfrak{U}^{w''}$ yields

$$[\mathcal{Y} : \mathcal{U}^{w''}] = \sum_{\mathfrak{C} \subset \overline{\mathfrak{Y}}} [\mathfrak{C} : \overline{\mathfrak{U}}^{w''}].$$

Since $\overline{\mathfrak{U}}^{w''} = (\overline{\mathbf{T}}'')^{w''}$ and $\text{in}_w(X) \cong \overline{D} \times (\overline{\mathbf{T}}')^{w''}$ for some dimension-zero scheme $\overline{D} \subset (\overline{\mathbf{T}}')^{w''}$ (cf. Remark 4.28), the reduced space underlying any irreducible component of $\text{in}_w(X)$ is isomorphic to $(\overline{\mathbf{T}}'')^{w''}$. Therefore $[\text{im}(\overline{\mathfrak{C}}) : \overline{\mathfrak{U}}^{w''}] = 1$ for any irreducible component $\overline{\mathfrak{C}} \subset \overline{\mathfrak{Y}}$, so $[\overline{\mathfrak{C}} : \overline{\mathfrak{U}}^{w''}] = [\overline{\mathfrak{C}} : \text{im}(\overline{\mathfrak{C}})]$ and the proposition follows. \blacksquare

Corollary 4.33. (Tropical projection formula) *In the situation of Theorem 4.31, the degree of $\psi^P : \mathcal{X}^P \rightarrow \mathcal{U}^{P''}$ is equal to $m_{\text{Trop}}(w)$ for any $w \in P$.*

Proof. Assuming that $P = w$ and $P'' = w''$ as in the proof of Proposition 4.32, the result follows immediately from Propositions 4.32 and 4.24. \blacksquare

Remark 4.34. The tropical projection formula is an equality of the degree of the morphism $\mathcal{X}^P \rightarrow \mathcal{U}^{P''}$ (a morphism on the generic fiber) with the degree of a morphism $\overline{\mathfrak{X}}^w \rightarrow \overline{\mathfrak{U}}^{w''}$ (a morphism on the special fiber). It is conceptually very close to the projection formula as stated in Proposition 3.32, as indeed that is the main tool used in its proof; it is for this reason that we call it the tropical projection formula.

5. THE STRUCTURE THEORY OF ANALYTIC CURVES

In order to make the results of the previous sections more precise and more explicit in the case of a curve X inside a torus, it is necessary to develop some of the structure theory of the analytification X^{an} . We will define the *skeleton* of X^{an} corresponding to a *semistable decomposition* of X^{an} into a disjoint union of open balls, punctured open balls, open annuli, and finitely many type-2 points, and explain how skeleta are related to semistable formal models. Almost all of the results along these lines are well-known to experts: most of the ideas go back to Berkovich, who originally introduced skeleta in [Ber90, Chapter 4] and [Ber99]; Thuillier also gives a related account of skeleta in his thesis [Thu05], and much of the content of this section is outlined without proof in [Tem]. We have chosen to give a relatively complete and self-contained exposition, because much of this material is difficult to extract from the current literature and essential for the rest of the paper. We also give an account of the metric structure on $\mathbf{H}(X^{\text{an}})$, which again has been known to experts for some time but for which there is no suitable reference in the generality that we require.

As always, we assume that K is an algebraically closed field that is complete with respect to a nontrivial nonarchimedean valuation.

5.1. Some analytic domains in \mathbf{A}^1 .

$$\text{trop} : \mathcal{M}(K[T]) = \mathbf{A}_{\text{an}}^1 \rightarrow \mathbf{R} \cup \{\infty\} \quad \text{is} \quad \text{trop}(\|\cdot\|) = -\log(\|T\|).$$

We use trop to define several analytic domains in \mathbf{A}_{an}^1 :

- For $a \in K^\times$ the *standard closed ball of radius $|a|$* is $\mathbf{B}(a) = \text{trop}^{-1}([\text{val}(a), \infty])$. (Note that this was denoted $\mathbf{B}(0, |a|)$ in (2.4).) This is a polyhedral domain whose ring of analytic functions is

$$K\langle a^{-1}t \rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n : |a_n| \cdot |a|^n \rightarrow 0 \text{ as } n \rightarrow \infty \right\}.$$

The supremum norm is given by

$$\left| \sum_{n=0}^{\infty} a_n t^n \right|_{\text{sup}} = \max \{ |a_n| \cdot |a|^n : n \geq 0 \}$$

and the canonical reduction is the polynomial ring $k[\tau]$ where τ is the residue of $a^{-1}t$.

- For $a \in K^\times$ the *standard open ball of radius $|a|$* is $\mathbf{B}(a)_+ = \text{trop}^{-1}((\text{val}(a), \infty])$. This is an open analytic domain which can be expressed as an increasing union of standard closed balls.
- For $a, b \in K^\times$ with $|a| \leq |b|$ the *standard closed annulus of inner radius $|a|$ and outer radius $|b|$* is $\mathbf{S}(a, b) = \text{trop}^{-1}([\text{val}(b), \text{val}(a)])$. This is a polytopal domain in \mathbf{G}_m^{an} (4.4); it is therefore an affinoid space whose ring of analytic functions is

$$K\langle at^{-1}, b^{-1}t \rangle = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n : |a_n| \cdot |a|^n \rightarrow 0 \text{ as } n \rightarrow +\infty, |a_n| \cdot |b|^n \rightarrow 0 \text{ as } n \rightarrow -\infty \right\}.$$

The supremum norm is given by

$$\left| \sum_{n=-\infty}^{\infty} a_n t^n \right|_{\text{sup}} = \max \{ |a_n| \cdot |a|^n, |a_n| \cdot |b|^n : n \in \mathbf{Z} \}$$

and the canonical reduction is $k[\sigma, \tau]/(\sigma\tau - \overline{a/b})$ where σ (resp. τ) is the residue of at^{-1} (resp. $b^{-1}t$) and $\overline{a/b} \in k$ is the residue of a/b . The canonical reduction is an integral domain if and only if $|a| = |b|$, in which case the supremum norm is multiplicative. The *(logarithmic) modulus* of $\mathbf{S}(a, b)$ is by definition $\text{val}(a) - \text{val}(b)$.

- In the above situation, if $|a| \leq 1$ and $|b| = 1$ we write $\mathbf{S}(a) := \mathbf{S}(a, 1) = \text{trop}^{-1}([0, \text{val}(a)])$. In this case

$$K\langle at^{-1}, t \rangle \cong K\langle s, t \rangle / (st - a).$$

- For $a, b \in K^\times$ with $|a| < |b|$ the *standard open annulus of inner radius $|a|$ and outer radius $|b|$* is $\mathbf{S}(a, b)_+ = \text{trop}^{-1}((\text{val}(b), \text{val}(a)))$. This is an open analytic domain which can be expressed as an increasing union of standard closed annuli. The *(logarithmic) modulus* of $\mathbf{S}(a, b)_+$ is by definition $\text{val}(a) - \text{val}(b)$. As above we write $\mathbf{S}(a)_+ := \mathbf{S}(a, 1)_+ = \text{trop}^{-1}((0, \text{val}(a)))$.
- For $a \in K^\times$ the *standard punctured open ball of radius $|a|$* is $\mathbf{S}(0, a)_+ = \text{trop}^{-1}((\text{val}(a), \infty))$, and the *standard punctured open ball of radius $|a|^{-1}$ around ∞* is $\mathbf{S}(a, \infty)_+ = \text{trop}^{-1}((-\infty, \text{val}(a)))$. These are open analytic domains which can be written as an increasing union of standard closed annuli. By convention we define the *modulus* of $\mathbf{S}(0, a)_+$ and $\mathbf{S}(a, \infty)_+$ to be infinity. We write $\mathbf{S}(0)_+ = \mathbf{S}(0, 1)_+$.

Note that if A is any of the above analytic domains in \mathbf{A}_{an}^1 then $A = \text{trop}^{-1}(\text{trop}(A))$. By a *standard generalized annulus* we will mean a standard closed annulus, a standard open annulus, or a standard punctured open ball, and by a *standard generalized open annulus* we will mean a standard open annulus or a standard punctured open ball. Note that by scaling we have isomorphisms

$$\mathbf{B}(a) \cong \mathbf{B}(1) \quad \mathbf{B}(a)_+ \cong \mathbf{B}(1)_+ \quad \mathbf{S}(a, b) \cong \mathbf{S}(ab^{-1}) \quad \mathbf{S}(a, b)_+ \cong \mathbf{S}(ab^{-1})_+ \quad \mathbf{S}(0, a)_+ \cong \mathbf{S}(0)_+$$

and taking $t \mapsto t^{-1}$ yields $\mathbf{S}(1, \infty)_+ \cong \mathbf{S}(0, 1)_+$.

Morphisms of standard closed annuli have the following structure:

Proposition 5.2. *Let $a \in R \setminus \{0\}$.*

- (1) *The units in $K\langle at^{-1}, t \rangle$ are the functions of the form*

$$(5.2.1) \quad f(t) = \alpha t^d (1 + g(t))$$

where $\alpha \in K^\times$, $d \in \mathbf{Z}$, and $|g|_{\text{sup}} < 1$.

- (2) *Let $f(t)$ be a unit as in (5.2.1) with $d > 0$ (resp. $d < 0$). The induced morphism $\varphi : \mathbf{S}(a) \rightarrow \mathbf{G}_m^{\text{an}}$ factors through a finite flat morphism $\mathbf{S}(a) \rightarrow \mathbf{S}(\alpha a^d, \alpha)$ (resp. $\mathbf{S}(a) \rightarrow \mathbf{S}(\alpha, \alpha a^d)$) of degree $|d|$.*
- (3) *Let $f(t)$ be a unit as in (5.2.1) with $d = 0$. The induced morphism $\varphi : \mathbf{S}(a) \rightarrow \mathbf{G}_m^{\text{an}}$ factors through a morphism $\mathbf{S}(a) \rightarrow \mathbf{S}(\alpha, \alpha)$ which is not finite.*

Proof. The first assertion is proved in [Thu05, Lemme 2.2.1] by considering the Newton polygon of $f(t)$. To prove (2) we easily reduce to the case $\alpha = 1$ and $d > 0$. Since $|f|_{\text{sup}} = 1$ and $|f^{-1}|_{\text{sup}} = |a|^{-d}$ the morphism φ factors set-theoretically through the affinoid domain $\mathbf{S}(a^d)$. Hence φ induces a morphism $\mathbf{S}(a) \rightarrow \mathbf{S}(a^d)$, so the homomorphism $K[s] \rightarrow K\langle at^{-1}, t \rangle$ extends to a homomorphism

$$F : K\langle a^d s^{-1}, s \rangle \longrightarrow K\langle at^{-1}, t \rangle \quad s \mapsto t^d (1 + g(t)), \quad a^d s^{-1} \mapsto (at^{-1})^d (1 + g(t))^{-1}.$$

Since $|g|_{\text{sup}} < 1$, the induced map on canonical reductions is

$$\tilde{F} : k[\sigma_1, \sigma_2] / (\sigma_1 \sigma_2 - \bar{a}^d) \longrightarrow k[\tau_1, \tau_2] / (\tau_1 \tau_2 - \bar{a}) \quad \sigma_i \mapsto \tau_i^d$$

where σ_1 (resp. σ_2, τ_1, τ_2) is the residue of $a^d s^{-1}$ (resp. s, at^{-1}, t). Now F is finite because \tilde{F} is finite [BGR84, Theorem 6.3.5/1], and F has degree d by the projection formula (3.32) because \tilde{F} has degree d on irreducible components. Flatness of F is automatic because its source and target are principal ideal domains: any affinoid algebra is noetherian, and if $\mathcal{M}(\mathcal{A})$ is an affinoid subdomain of $\mathbf{A}_{\text{an}}^1 = \text{Spec}(K[t])^{\text{an}}$ then any maximal ideal of \mathcal{A} is the extension of a maximal ideal of $K[t]$ by [Con99, Lemma A.1.2(1)].

For (3), as above φ factors through $\mathbf{S}(1, 1)$ if we assume $\alpha = 1$, so we get a homomorphism $F : K\langle a^d s^{-1}, s \rangle \rightarrow K\langle t, t^{-1} \rangle$. In this case the map \tilde{F} on canonical reductions is clearly not finite, so F is not finite. \blacksquare

5.3. The skeleton of a standard generalized annulus. Define a section $\sigma : \mathbf{R} \rightarrow \mathbf{G}_m^{\text{an}}$ of the tropicalization map $\text{trop} : \mathbf{G}_m^{\text{an}} \rightarrow \mathbf{R}$ by

$$(5.3.1) \quad \sigma(r) = \|\cdot\|_r \quad \text{where} \quad \left\| \sum_{n=-\infty}^{\infty} a_n t^n \right\|_r = \max \{ |a_n| \cdot \exp(-rn) : n \in \mathbf{Z} \}.$$

When $r \in G$ the point $\sigma(r)$ is the Shilov boundary point of the (strictly) affinoid domain $\text{trop}^{-1}(r)$, and when $r \notin G$ we have $\text{trop}^{-1}(r) = \{\sigma(r)\}$. The map σ is easily seen to be continuous. We restrict σ to obtain continuous sections

$$\begin{aligned} [\text{val}(b), \text{val}(a)] &\longrightarrow \mathbf{S}(a, b) & (\text{val}(b), \text{val}(a)) &\longrightarrow \mathbf{S}(a, b)_+ \\ (\text{val}(a), \infty) &\longrightarrow \mathbf{S}(0, a)_+ & (-\infty, \text{val}(a)) &\longrightarrow \mathbf{S}(a, \infty)_+ \end{aligned}$$

of trop .

Definition. Let A be a standard generalized annulus. The *skeleton* of A is the closed subset

$$\Sigma(A) := \sigma(\mathbf{R}) \cap A = \sigma(\text{trop}(A)).$$

More explicitly, the skeleton of $\mathbf{S}(a, b)$ (resp. $\mathbf{S}(a, b)_+$, resp. $\mathbf{S}(0, a)_+$, resp. $\mathbf{S}(a, \infty)_+$) is

$$\begin{aligned} \Sigma(\mathbf{S}(a, b)) &:= \sigma(\mathbf{R}) \cap \mathbf{S}(a, b) = \sigma([\text{val}(b), \text{val}(a)]) \\ \Sigma(\mathbf{S}(a, b)_+) &:= \sigma(\mathbf{R}) \cap \mathbf{S}(a, b)_+ = \sigma((\text{val}(b), \text{val}(a))) \\ \Sigma(\mathbf{S}(0, a)_+) &:= \sigma(\mathbf{R}) \cap \mathbf{S}(0, a)_+ = \sigma((\text{val}(a), \infty)) \\ \Sigma(\mathbf{S}(a, \infty)_+) &:= \sigma(\mathbf{R}) \cap \mathbf{S}(a, \infty)_+ = \sigma((-\infty, \text{val}(a))). \end{aligned}$$

We identify $\Sigma(A)$ with $\text{trop}(A)$ via trop or σ .

Note that $\tau_A := \sigma \circ \text{trop}$ is a retraction of a standard generalized annulus A onto its skeleton. This can be shown to be a strong deformation retraction [Ber90, Proposition 4.1.6]. Note also that the length of the skeleton of a standard generalized annulus is equal to its modulus.

The set-theoretic skeleton has the following intrinsic characterization:

Proposition 5.4. ([Thu05, Proposition 2.2.5]) *The skeleton of a standard generalized annulus is the set of all points that do not admit an affinoid neighborhood isomorphic to $\mathbf{B}(1)$.*

The skeleton behaves well with respect to maps between standard generalized annuli:

Proposition 5.5. *Let A be a standard generalized annulus of nonzero modulus and let $\varphi : A \rightarrow \mathbf{G}_m^{\text{an}}$ be a morphism. Suppose that $\text{trop} \circ \varphi : \Sigma(A) \rightarrow \mathbf{R}$ is not constant. Then:*

- (1) *For $x \in \Sigma(A)$ we have*

$$\text{trop} \circ \varphi(x) = d \text{trop}(x) + \text{val}(\alpha)$$

for some nonzero integer d and some $\alpha \in K^\times$.

- (2) *Let $B = \varphi(A)$. Then $B = \text{trop}^{-1}(\text{trop}(\varphi(A)))$ is a standard generalized annulus in \mathbf{G}_m^{an} of the same type, and $\varphi : A \rightarrow B$ is a finite morphism of degree $|d|$.*

- (3) *$\varphi(\Sigma(A)) = \Sigma(B)$ and the following square commutes:*

$$\begin{array}{ccc} \text{trop}(A) & \xrightarrow{d(\cdot) + \text{val}(\alpha)} & \text{trop}(B) \\ \sigma \downarrow & & \downarrow \sigma \\ \Sigma(A) & \xrightarrow{\varphi} & \Sigma(B) \end{array}$$

Proof. Let $A' \cong \mathbf{S}(a) \subset A$ be a standard closed annulus of nonzero modulus such that $\text{trop} \circ \varphi$ is not constant on $\Sigma(A')$. The morphism φ is determined by a unit $f \in K\langle at^{-1}, t \rangle^\times$, and for $x \in \Sigma(A')$ we have $\text{trop}(\varphi(x)) = -\log |f(x)|$. Writing $f(t) = \alpha t^d(1 + g(t))$ as in (5.2.1), if $r = \text{trop}(x)$ then $-\log |f(x)| = -\log \|f\|_r = dr + \text{val}(\alpha)$ since $\|1 + g\|_r = 1$. Since $\text{trop} \circ \varphi$ is nonconstant on $\Sigma(A')$ we must have $d \neq 0$. Part (1) follows by writing A as an increasing union of standard closed annuli and applying the same argument. The equality $B = \text{trop}^{-1}(\text{trop}(\varphi(A)))$ follows from Proposition 5.2(2) in the same way; since $\text{trop}(\varphi(A))$ is a closed interval (resp. open interval, resp. open ray) when $\text{trop}(A)$ is a closed interval (resp. open interval, resp. open ray), it follows that B is a standard generalized annulus of the same type as A .

For part (3) it suffices to show that $\varphi(\sigma(r)) = \sigma(dr + \text{val}(\alpha))$ for $r \in \text{trop}(A)$. This follows from the above because $\sigma(dr + \text{val}(\alpha))$ is the supremum norm on $\text{trop}^{-1}(dr + \text{val}(\alpha))$ (when $r \in G$) and φ maps $\text{trop}^{-1}(r)$ surjectively onto $\text{trop}^{-1}(dr + \text{val}(\alpha))$. \blacksquare

Corollary 5.6. *Let $\varphi : A_1 \rightarrow A_2$ be a finite morphism of standard generalized annuli and let d be the degree of φ . Then $\varphi(\Sigma(A_1)) = \Sigma(A_2)$, $\varphi(\sigma(r)) = \sigma(\pm dr + \text{val}(\alpha))$ for all $r \in \text{trop}(A_1)$ and some $\alpha \in K^\times$, and the modulus of A_2 is d times the modulus of A_1 . In particular, two standard generalized annuli of the same type are isomorphic if and only if they have the same modulus.*

Proof. If the modulus of A_1 is zero then the result follows easily from Proposition 5.2. Suppose that the modulus of A_1 is nonzero. By Proposition 5.5, the only thing to show is that $\text{trop} \circ \varphi$ is not constant on $\Sigma(A)$. This is an immediate consequence of Proposition 5.2(3). \blacksquare

5.7. General annuli and balls. In order to distinguish the properties of a standard generalized annulus and its skeleton that are invariant under isomorphism, it is convenient to make the following definition.

Definition. A *closed ball* (resp. *closed annulus*, resp. *open ball*, resp. *open annulus*, resp. *punctured open ball*) is a K -analytic space isomorphic to a standard closed ball (resp. standard closed annulus, resp. standard open ball, resp. standard open annulus, resp. standard punctured open ball). A *generalized annulus* is a closed annulus, an open annulus, or a punctured open ball, and a *generalized open annulus* is an open annulus or a punctured open ball.

5.8. Let A be a generalized annulus and fix an isomorphism $\varphi : A \xrightarrow{\sim} A'$ with a standard generalized annulus A' . The *skeleton* of A is defined to be $\Sigma(A) := \varphi^{-1}(\Sigma(A'))$. By Proposition 5.4 (or Corollary 5.6) this is a well-defined closed subset of $\Sigma(A)$. We will view $\Sigma(A)$ as a closed interval (resp. open interval, resp. open ray) with endpoints in G , well-defined up to affine transformations of the form $r \mapsto \pm r + \text{val}(\alpha)$ for $\alpha \in K^\times$. In particular $\Sigma(A)$ is naturally a metric space, and it makes sense to talk about piecewise affine-linear functions on $\Sigma(A)$ and of the slope of a linear function on $\Sigma(A)$ up to sign.

The retraction $\tau_{A'} = \sigma \circ \text{trop} : A' \rightarrow \Sigma(A')$ induces a retraction $\tau_A : A \rightarrow \Sigma(A)$. By Proposition 5.5 this retraction is also independent of the choice of A' .

Definition 5.9. Let A be a generalized annulus, an open ball, or a closed ball. A *meromorphic function* on A is by definition a quotient of an analytic function on A by a nonzero analytic function on A .

Note that a meromorphic function f on A is an analytic function defined on the open analytic domain of A obtained by deleting the poles of f . If A is affinoid then f has only finitely many poles.

Let A be a generalized annulus, let $F : \Sigma(A) \rightarrow \mathbf{R}$ be a piecewise linear function, and let x be contained in the interior of $\Sigma(A)$. The *change of slope* of F at x is defined to be

$$\lim_{\varepsilon \rightarrow 0} (F'(x + \varepsilon) - F'(x - \varepsilon));$$

this is independent of the choice of identification of $\Sigma(A)$ with an interval in \mathbf{R} .

We will need the following special case of the Slope Formula (5.69). Its proof is an easy Newton polygon computation.

Proposition 5.10. *Let A be a generalized annulus, let f be a meromorphic function on A , and define $F : \Sigma(A) \rightarrow \mathbf{R}$ by $F(x) = -\log |f(x)|$.*

- (1) *F is a piecewise linear function with integer slopes, and for x in the interior of $\Sigma(A)$ the change of slope of F at x is equal to the number of poles of f retracting to x minus the number of zeros of f retracting to x , counted with multiplicity.*
- (2) *Suppose that $A = \mathbf{S}(0)_+$ and that f extends to a meromorphic function on $\mathbf{B}(1)_+$. Then for all $r \in (0, \infty)$ such that $r > \text{val}(y)$ for all zeros and poles y of f in A , we have $F'(r) = \text{ord}_0(f)$.*

Corollary 5.11. *Let f be an analytic function on $\mathbf{S}(0)_+$ that extends to a meromorphic function on $\mathbf{B}(1)_+$ with a pole at 0 of order d . Suppose that f has fewer than d zeros on $\mathbf{S}(0)_+$. Then $F = \log |f|$ is a monotonically increasing function on $\Sigma(\mathbf{S}(0)_+) = (0, \infty)$.*

The following facts will also be useful:

Lemma 5.12. *Let A be a generalized annulus. Then the open analytic domain $A \setminus \Sigma(A)$ is isomorphic to an infinite disjoint union of open balls. Each connected component B of $A \setminus \Sigma(A)$ retracts onto a single point $x \in \Sigma(A)$, and the closure of B in A is equal to $B \cup \{x\}$.*

Proof. First we assume that A is the standard closed annulus $\mathbf{S}(1) = \mathcal{M}(K\langle t^{\pm 1} \rangle)$ of modulus zero. Then $\Sigma(A) = \{x\}$ is the Shilov boundary point of A . The canonical reduction of A is isomorphic to $\mathbf{G}_{m,k}$, the inverse image of the generic point of $\mathbf{G}_{m,k}$ is x (3.8), the inverse image of a residue class $\bar{y} \in k^\times = \mathbf{G}_m(k)$ is the open ball $\{\|\cdot\| : \|t - y\| < 1\}$ (where $y \in R^\times$ reduces to \bar{y}), and the fibers over the closed points of $\mathbf{G}_{m,k}$ are the connected components of $A \setminus \{x\}$ by [Thu05, Lemme 2.1.13]. This proves the first assertion, and the second follows from the anti-continuity of the reduction map.

Now let A be any generalized annulus; we may assume that A is standard. Let $r \in \text{trop}(A)$. If $r \notin G$ then $\text{trop}^{-1}(r)$ is a single point of type 3, so suppose $r \in G$, say $r = \text{val}(a)$ for $a \in K^\times$. After translating by a^{-1} we may and do assume that $r = 0$, so $\text{trop}^{-1}(r) = \mathbf{S}(1) = \mathcal{M}(K\langle t^{\pm 1} \rangle)$. The subset $\mathbf{S}(1) \setminus \{\sigma(0)\}$ is clearly closed in $A \setminus \Sigma(A)$, and it is open as well since it is the union of the open balls $\{\|\cdot\| : \|t - y\| < 1\}$ for $y \in R^\times$. Therefore the connected components of $\mathbf{S}(1) \setminus \{\sigma(0)\}$ are also connected components of $A \setminus \Sigma(A)$, so we are reduced to the case treated above. ■

Lemma 5.13. *Let A be a generalized annulus and let f be a unit on A . Then $x \mapsto \log |f(x)|$ factors through the retraction $\tau_A : A \rightarrow \Sigma(A)$. In particular, $x \mapsto \log |f(x)|$ is locally constant away from $\Sigma(A)$.*

Proof. This follows immediately from Lemma 5.12 and the elementary fact that a unit on an open ball has constant absolute value. ■

5.14. Semistable decompositions and skeleta of curves. For the rest of this section X denotes a smooth connected algebraic curve over K , \widehat{X} denotes its smooth completion, and $D = \widehat{X} \setminus X$ denotes the set of punctures. We will define a skeleton inside of X relative to the following kind of decomposition of X :

Definition 5.15. A *semistable vertex set* of \widehat{X} is a finite set V of type-2 points of \widehat{X}^{an} such that $\widehat{X}^{\text{an}} \setminus V$ is a disjoint union of open balls and finitely many open annuli. A *semistable vertex set* of X is a semistable vertex set of \widehat{X} such that every puncture in D is contained in a connected component of $\widehat{X}^{\text{an}} \setminus V$ isomorphic to an open ball. A decomposition of X^{an} into a semistable vertex set and a disjoint union of open balls and finitely many generalized open annuli is called a *semistable decomposition* of X .

When we refer to ‘an open ball in a semistable decomposition of X ’ or ‘a generalized open annulus in a semistable decomposition of X ’ we will always mean a connected component of $X^{\text{an}} \setminus V$ of the specified type. Note that the punctured open balls in a semistable decomposition of X are in bijection with D , and that there are no punctured open balls in a semistable decomposition of a complete curve. A semistable vertex set of X is also a semistable vertex set of \widehat{X} .

Lemma 5.16. *Let V be a semistable vertex set of X , let A be a connected component of $X^{\text{an}} \setminus V$, and let \overline{A} be the closure of A in \widehat{X}^{an} . Let $\partial_{\text{lim}} A = \overline{A} \setminus A$ be the limit boundary of A , i.e. the set of limit points of A in \widehat{X}^{an} that are not contained in A .⁷*

- (1) *If A is an open ball then $\partial_{\text{lim}} A = \{x\}$ for some $x \in V$.*
- (2) *Suppose that A is an open annulus, and fix an isomorphism $A \cong \mathbf{S}(a)_+$. Let $r = \text{val}(a)$. Then $\sigma : (0, r) \rightarrow A$ extends in a unique way to a continuous map $\sigma : [0, r] \rightarrow X^{\text{an}}$ such that $\sigma(0), \sigma(r) \in V$, and $\partial_{\text{lim}} A = \{\sigma(0), \sigma(r)\}$. (It may happen that $\sigma(0) = \sigma(r)$.)*
- (3) *Suppose that A is a punctured open ball, and fix an isomorphism $A \cong \mathbf{S}(0)_+$. Then $\sigma : (0, \infty) \rightarrow A$ extends in a unique way to a continuous map $\sigma : [0, \infty] \rightarrow \widehat{X}^{\text{an}}$ such that $\sigma(0) \in V$, $\sigma(\infty) \in D$, and $\partial_{\text{lim}} A = \{\sigma(0), \sigma(\infty)\}$.*

Proof. First note that in (1) and (2), \overline{A} is the closure of A in X^{an} because every point of $\widehat{X} \setminus X$ has an open neighborhood disjoint from A . Since A is closed in $X \setminus V$, its limit boundary is contained in V .

⁷As opposed to the canonical boundary discussed in [Ber90, §2.5.7].

Suppose that A is an open ball, and fix an isomorphism $\varphi : \mathbf{B}(1)_+ \xrightarrow{\sim} A$. For $r \in (0, \infty)$ we define $\| \cdot \|_r \in \mathbf{B}(1)_+$ by (5.3.1). Fix an affine open subset X' of X such that $A \subset (X')^{\text{an}}$. For any $f \in K[X']$ the map $r \mapsto \log \|f\|_r$ is piecewise linear with finitely many changes in slope by Proposition 5.10. Therefore we may define $\|f\| = \lim_{r \rightarrow 0} \|f\|_r \in \mathbf{R}$. The map $f \mapsto \|f\|$ is easily seen to be a multiplicative norm on $K[X']$, hence defines a point $x \in (X')^{\text{an}} \subset X^{\text{an}}$.

Let y be the Shilov point of $\mathbf{B}(1)$ and let $A' = \mathbf{B}(1)_+ \cup \{y\}$. Since $\mathbf{B}(1) \setminus \{y\}$ is a disjoint union of open balls it is clear that A' is a closed, hence compact subset of $\mathbf{B}(1)$. Extend φ to a map $A' \rightarrow (X')^{\text{an}} \subset X^{\text{an}}$ by $\varphi(y) = x$. We claim that φ is continuous. By the definition of the topology on $(X')^{\text{an}}$ it suffices to show that the set $U = \{z \in A' : |f(\varphi(z))| \in (c_1, c_2)\}$ is open for all $f \in K[X']$ and all $c_1 < c_2$. Since $U \cap \mathbf{B}(1)_+$ is open, we need to show that U contains a neighborhood of y if $y \in U$, i.e., if $\|f\| \in (c_1, c_2)$. Choose $a \in \mathfrak{m}_R \setminus \{0\}$ such that f has no zeros in $\mathbf{S}(a)_+$. Note that $\mathbf{S}(a)_+ \cup \{y\}$ is a neighborhood of y in A' . Since $\|f\| = \lim_{r \rightarrow 0} \|f\|_r$ we have that $\|f\|_r \in (c_1, c_2)$ for r close enough to 1; hence we may shrink $\mathbf{S}(a)_+$ so that $\varphi(\Sigma(\mathbf{S}(a)_+)) \subset (r_1, r_2)$. With Lemma 5.13 this implies that $\mathbf{S}(a)_+ \subset U$, so φ is indeed continuous. Since A' is compact we have that $\varphi(A') = A \cup \{x\}$ is closed, which completes the proof of (1).

If $A \cong \mathbf{S}(0)_+$ is a punctured open ball then certainly the puncture 0 is in \overline{A} . The above argument effectively proves the rest of (3), and (2) is proved in exactly the same way. \blacksquare

Definition 5.17. Let V be a semistable vertex set of X . The *skeleton* of X with respect to V is

$$\Sigma(X, V) = V \cup \bigcup \Sigma(A)$$

where A runs over all of the connected components of $X^{\text{an}} \setminus V$ that are generalized open annuli.

Lemma 5.18. Let V be a semistable vertex set of X and let $\Sigma = \Sigma(X, V)$ be the associated skeleton. Then:

- (1) Σ is a closed subset of X^{an} which is compact if and only if $X = \widehat{X}$.
- (2) The limit boundary of Σ in \widehat{X}^{an} is equal to D .
- (3) The connected components of $X^{\text{an}} \setminus \Sigma(X, V)$ are open balls, and the limit boundary $\partial_{\text{lim}} B$ of any connected component B is a single point $x \in \Sigma(X, V)$.
- (4) Σ is equal to the set of points in X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbf{B}(1)$ and disjoint from V .

Proof. The first two assertions are clear from Lemma 5.16, and the third follows from Lemmas 5.16 and 5.12. Let Σ' be the set of points in X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbf{B}(1)$ and disjoint from V . We have $\Sigma' \subset \Sigma$ by (3). For the other inclusion, let $x \in \Sigma$. If $x \in V$ then clearly $x \in \Sigma'$, so suppose $x \notin V$. Then the connected component A of x in $X^{\text{an}} \setminus V$ is a generalized open annulus; since any connected neighborhood of x is contained in A , we have $x \in \Sigma'$ by Proposition 5.4. \blacksquare

Definition 5.19. Let V be a semistable vertex set of X . The *completed skeleton* of X with respect to V is defined to be the closure of $\Sigma(X, V)$ in \widehat{X}^{an} and is denoted $\widehat{\Sigma}(X, V)$, so $\widehat{\Sigma}(X, V) = \Sigma(X, V) \cup D$. The completed skeleton has the structure of a graph with vertices $V \cup D$; the interiors of the edges of $\Sigma(X, V)$ are the skeleta of the generalized open annuli in the semistable decomposition of X coming from V . We say that V is *strongly semistable* if the graph $\widehat{\Sigma}(X, V)$ has no loop edges.

Remark 5.20. By Lemma 5.18(1), if $X = \widehat{X}$ then the skeleton $\Sigma(X, V) = \widehat{\Sigma}(X, V)$ is a finite metric graph (cf. (5.36)). If X is not proper then $\widehat{\Sigma}(X, V)$ is a finite graph with vertex set $V \cup D$, but it is not a metric graph since it has edges of infinite length.

Definition 5.21. Let V be a semistable vertex set of X and let $\Sigma = \Sigma(X, V)$. We define a retraction $\tau_V = \tau_\Sigma : X^{\text{an}} \rightarrow \Sigma$ as follows. Let $x \in X^{\text{an}} \setminus \Sigma$ and let B_x be the connected component of x in $X^{\text{an}} \setminus \Sigma$. Then $\partial_{\text{lim}}(B_x) = \{y\}$ for a single point $y \in X^{\text{an}}$; we set $\tau_V(x) = y$.

Lemma 5.22. *Let V be a semistable vertex set of X . The retraction $\tau_V : X^{\text{an}} \rightarrow \Sigma(X, V)$ is continuous, and if A is a generalized open annulus in the semistable decomposition of X then τ_V restricts to the retraction $\tau_A : A \rightarrow \Sigma(A)$ defined in (5.8).*

Proof. The second assertion follows from Lemma 5.12, so τ_V is continuous when restricted to any connected component A of $X^{\text{an}} \setminus V$ which is a generalized open annulus. Hence it is enough to show that if $x \in V$ and U is an open neighborhood of x then $\tau_V^{-1}(U)$ contains an open neighborhood of x . This is left as an exercise to the reader. ■

Proposition 5.23. *Let V be a semistable vertex set of X . Then $\Sigma(X, V)$ and $\widehat{\Sigma}(X, V)$ are connected.*

Proof. This follows from the continuity of τ_V and the connectedness of X^{an} . ■

The skeleton of a curve naturally carries the following kind of combinatorial structure, which is similar to that of a metric graph.

Definition 5.24. A dimension-1 abstract G -affine polyhedral complex is a combinatorial object Σ consisting of the following data. We are given a finite discrete set V of vertices and a collection of finitely many segments and rays, where a segment is a closed interval in \mathbf{R} with distinct endpoints in G and a ray is a closed ray in \mathbf{R} with endpoint in G . Segments and rays are only defined up to isometries of \mathbf{R} of the form $r \mapsto \pm r + \alpha$ for $\alpha \in G$. The segments and rays are collectively called edges of Σ . Finally, we are given an identification of the endpoints of the edges of Σ with vertices. The complex Σ has an obvious realization as a topological space, which we will also denote by Σ . If Σ is connected then it is a metric space under the shortest-path metric.

A morphism of dimension-1 abstract G -affine polyhedral complexes is a continuous function $\varphi : \Sigma \rightarrow \Sigma'$ sending vertices to vertices and such that if $e \subset \Sigma$ is an edge then either $\varphi(e)$ is a vertex of Σ' , or $\varphi(e)$ is an edge of Σ' and for all $r \in e$ we have $\varphi(r) = dr + \alpha$ for a nonzero integer d and some $\alpha \in G$.

A refinement of a dimension-1 abstract G -affine polyhedral complex is a complex Σ' obtained from Σ by inserting vertices at G -points of edges of Σ and dividing those edges in the obvious way. Note that Σ and Σ' have the same topological and metric space realizations.

Remark 5.25. Abstract integral G -affine polyhedral complexes of arbitrary dimension are defined in [Thu05, § 1] in terms of rings of integer-slope G -affine functions. In the one-dimensional case the objects of loc. cit. are roughly the same as the dimension-1 abstract integral G -affine polyhedral complexes in the sense of our ad-hoc definition above, since the knowledge of what functions on a line segment have slope one is basically the same as the data of a metric. We choose to use this definition for concreteness and in order to emphasize the metric nature of these objects.

5.26. Let V be a semistable vertex set of X . Then $\Sigma(X, V)$ is a dimension-1 abstract G -affine polyhedral complex with vertex set V whose edges are the closures of the skeleta of the generalized open annuli in the semistable decomposition of X . In particular, $\Sigma(X, V)$ is a metric space, and each edge e of $\Sigma(X, V)$ is identified via a local isometry with the skeleton of the corresponding generalized open annulus. Note that if e is a segment then the length of e is equal to the modulus of the corresponding open annulus. The G -points of $\Sigma(X, V)$ are exactly the type-2 points of X contained in $\Sigma(X, V)$.

Proposition 5.27. *Let V be a semistable vertex set of X and let X' be a nonempty open subscheme of X .*

- (1) *Let V' be a semistable vertex set of X' containing V . Then $\Sigma(X, V) \subset \Sigma(X', V')$ and $\Sigma(X', V')$ induces a refinement of $\Sigma(X, V)$. Furthermore, $\tau_{\Sigma(X, V)} \circ \tau_{\Sigma(X', V')} = \tau_{\Sigma(X, V)}$.*
- (2) *Let $V' \subset \Sigma(X, V)$ be a finite set of type-2 points. Then $V \cup V'$ is a semistable vertex set of X and $\Sigma(X, V \cup V')$ is a refinement of $\Sigma(X, V)$.*
- (3) *Let $W \subset X^{\text{an}}$ be a finite set of type-2 points. Then there is a semistable vertex set V' of X' containing $V \cup W$.*

Proof. In (1), the inclusion $\Sigma(X, V) \subset \Sigma(X', V')$ follows from Lemma 5.18(4), and the fact that $\Sigma(X', V')$ induces a refinement of $\Sigma(X, V)$ is an easy consequence of the structure of morphisms of

generalized open annuli (Proposition 5.5). The equality $\tau_{\Sigma(X,V)} \circ \tau_{\Sigma(X',V')} = \tau_{\Sigma(X,V)}$ follows from the definitions. In (2) we may assume that V' is a single point; it then suffices to show that if A is a generalized open annulus and $x \in \Sigma(A)$ is a type-2 point then $A \setminus \{x\}$ is a disjoint union of generalized open annuli and open balls. Choose an identification of A with a standard generalized open annulus $\mathbf{S}(a, b)_+$ such that $0 \in (\text{trop}(b), \text{trop}(a))$ and $x = \sigma(0)$. As in the proof of Lemma 5.12 we have that $\mathbf{S}(1) \setminus \{x\}$ is a disjoint union of open balls, and is open and closed in A ; hence

$$A \setminus \{x\} = \mathbf{S}(a, 1)_+ \amalg (\mathbf{S}(1) \setminus \{x\}) \amalg \mathbf{S}(1, b)_+$$

is a disjoint union of generalized open annuli and open balls.

It suffices to prove (3) when $W = \{x\}$ and $X = X'$, and when $W = \emptyset$ and $X \setminus X' = \{y\}$. In the first case, we may assume that $x \notin \Sigma(X, V)$ by the above. Suppose that the connected component A of $X^{\text{an}} \setminus V$ containing x is an open ball. One shows as in (2) that $A \setminus \{x\}$ is a disjoint union of open balls and an open annulus, so $V \cup \{x\}$ is a semistable vertex set. If A is a generalized open annulus then the connected component of x in $A \setminus \{\tau_A(x)\}$ is an open ball, so $V \cup \{x, \tau_A(x)\}$ is a semistable vertex set. In the case $X \setminus X' = \{y\}$ one proceeds in exactly the same way. ■

Corollary 5.28. *Any semistable vertex set of X is contained in a strongly semistable vertex set of X .*

5.29. Semistable models. To give a semistable vertex set of the complete curve \widehat{X} is basically equivalent to constructing a semistable (formal) model of \widehat{X} . The formal semistable reduction theory of a smooth complete algebraic curve was worked out carefully in [BL85] in the language of rigid analytic spaces and formal analytic varieties (see Remark 5.30(3)); one can view much of this section as a translation of that paper into our language.

Definition.

- (1) A connected reduced algebraic curve over a field k is called *semistable* if its singularities are ordinary double points. It is *strongly semistable* if in addition its irreducible components are smooth.
- (2) A (strongly) *semistable formal R -curve* is an integral admissible formal R -curve \mathfrak{X} whose special fiber is a (strongly) semistable curve. A (strongly) *semistable formal model* for \widehat{X} is a (strongly) semistable proper formal R -curve with analytic generic fiber isomorphic to \widehat{X}^{an} .

Remark 5.30.

- (1) Let \mathfrak{X} be a (strongly) semistable formal R -curve. Then \mathfrak{X} is proper if and only if \mathfrak{X}^{an} is proper by [Tem00, Corollary 4.4]. Therefore the properness hypothesis in the definition of a (strongly) semistable formal model for \widehat{X} is redundant.
- (2) A (strongly) *semistable algebraic model* for \widehat{X} is a flat and integral proper relative curve $\mathcal{X} \rightarrow \text{Spec}(R)$ whose special fiber is a (strongly) semistable curve and whose generic fiber is isomorphic to \widehat{X} . A (strongly) semistable algebraic model \mathcal{X} for \widehat{X} gives rise to a (strongly) semistable formal model \mathfrak{X} for \widehat{X} by ϖ -adically completing. Indeed, \mathfrak{X} and \mathcal{X} have the same special fiber, and $\mathfrak{X}^{\text{an}} \cong \widehat{X}^{\text{an}}$ by [Con99, Theorem A.3.1(4)]. Conversely, a (strongly) semistable formal model for \widehat{X} uniquely algebraizes to a (strongly) semistable algebraic model by a suitable formal GAGA theorem over R [Abb10, Corollaire 2.3.19]. Hence there is no essential difference between the algebraic and formal semistable reduction theories of \widehat{X} . We will generally work with algebraic semistable models in examples.
- (3) Let \mathfrak{X} be a semistable formal R -curve. Since $\overline{\mathfrak{X}}$ is reduced, \mathfrak{X} is a formal analytic variety in the sense of (3.5) or [BL85]. In particular, if $\text{Spf}(A)$ is a formal affine open subset of \mathfrak{X} then A is the ring of power-bounded elements of $A \otimes_R K$ and $A \otimes_R k$ is the canonical model of $A \otimes_R K$.

Let $a \in R \setminus \{0\}$. The *standard formal annulus of modulus* $\text{val}(a)$ is defined to be

$$\mathfrak{S}(a) := \text{Spf}(R\langle s, t \rangle / (st - a)).$$

This is the canonical model of the standard closed annulus $\mathbf{S}(a)$.

Proposition 5.31. *Let \mathfrak{X} be a strongly semistable formal R -curve and let $\xi \in \overline{\mathfrak{X}}$ be a singular point of $\overline{\mathfrak{X}}$. There is a formal neighborhood \mathfrak{U} of ξ and an étale morphism $\varphi : \mathfrak{U} \rightarrow \mathfrak{S}(a)$ for some $a \in \mathfrak{m}_R \setminus \{0\}$ such that φ^{an} restricts to an isomorphism $\text{red}^{-1}(\xi) \xrightarrow{\sim} \mathfrak{S}(a)_+$.*

Proof. This is essentially [BL85, Proposition 2.3]; here we explain how the proof of loc. cit. implies the proposition. Shrinking \mathfrak{X} if necessary, we may and do assume that $\mathfrak{X} = \text{Spf}(A)$ is affine and connected, and that the maximal ideal $\mathfrak{m}_\xi \subset \overline{A} = A \otimes_R k$ corresponding to ξ is generated by two functions $\overline{f}, \overline{g} \in \overline{A}$ whose product is zero (this is possible because \mathfrak{X} is strongly semistable). Let $k[x, y]/(xy) \rightarrow \overline{A}$ be the homomorphism sending $x \mapsto \overline{f}$ and $y \mapsto \overline{g}$. Since ξ is an ordinary double point, we can choose \overline{f} and \overline{g} such that the map on completed local rings $k[[x, y]]/(xy) \rightarrow \widehat{\mathcal{O}}_{\overline{\mathfrak{X}}, \xi}$ is an isomorphism. It follows from [EGAIV₄, Prop. 17.6.3] that the morphism $\text{Spec}(\overline{A}) \rightarrow \text{Spec}(k[x, y]/(xy))$ is étale at ξ , so shrinking \mathfrak{X} further we may assume that $\text{Spec}(\overline{A}) \rightarrow \text{Spec}(k[x, y]/(xy))$ is étale. One then proceeds as in the proof of [BL85, Proposition 2.3] to find lifts $f, g \in A$ of $\overline{f}, \overline{g}$ such that $fg = a \in R \setminus \{0\}$; the induced morphism $\mathfrak{X} \rightarrow \mathfrak{S}(a)$ is étale because it lifts an étale morphism on the special fiber. The fact that φ restricts to an isomorphism $\text{red}^{-1}(\xi) \xrightarrow{\sim} \mathfrak{S}(a)_+$ is part (i) of loc. cit. \blacksquare

The following characterization of strongly semistable formal R -curves is also commonly used in the literature, for example in [Thu05, Définition 2.2.8] (see also Remarque 2.2.9 in loc. cit.).

Corollary 5.32. *An integral admissible formal R -curve \mathfrak{X} is strongly semistable if and only if it has a covering by Zariski-open sets \mathfrak{U} which admit an étale morphism to $\mathfrak{S}(a_{\mathfrak{U}})$ for some $a_{\mathfrak{U}} \in R \setminus \{0\}$.*

Let $\mathfrak{X}, \mathfrak{X}'$ be two semistable formal models for \widehat{X} . We say that \mathfrak{X} *dominates* \mathfrak{X}' , and we write $\mathfrak{X} \geq \mathfrak{X}'$, if there exists an R -morphism $\mathfrak{X} \rightarrow \mathfrak{X}'$ inducing the identity on the generic fiber \widehat{X}^{an} . Such a morphism is unique if it exists. The relation \geq is a partial ordering on the set of semistable formal models for \widehat{X} . (We will always consider semistable formal models of \widehat{X} up to isomorphism; any isomorphism is unique.)

5.33. Semistable models and semistable decompositions. The special fiber of the canonical model for $\mathbf{B}(1)$ is isomorphic to A_k^1 , and the inverse image of the origin is the open unit ball $\mathbf{B}(1)_+$. When $|a| < 1$ the special fiber of $\mathfrak{S}(a)$ is isomorphic to $k[x, y]/(xy)$, and the inverse image of the origin under the reduction map is $\mathfrak{S}(a)_+$. The following much stronger version of these facts provides the relation between semistable models and semistable decompositions of \widehat{X} .

Theorem 5.34. (Berkovich, Bosch-Lütkebohmert) *Let \mathfrak{X} be an integral admissible formal R -curve with reduced special fiber and let $\xi \in \overline{\mathfrak{X}}$ be any point.*

- (1) ξ is a generic point if and only if $\text{red}^{-1}(\xi)$ is a single type-2 point of \mathfrak{X}^{an} .
- (2) ξ is a smooth closed point if and only if $\text{red}^{-1}(\xi) \cong \mathbf{B}(1)_+$.
- (3) ξ is an ordinary double point if and only if $\text{red}^{-1}(\xi) \cong \mathfrak{S}(a)_+$ for some $a \in \mathfrak{m}_R \setminus \{0\}$.

Proof. As in Remark 5.30(3) the hypothesis on the special fiber of \mathfrak{X} allows us to view \mathfrak{X} as a formal analytic variety. Hence the first statement follows from [Ber90, Proposition 2.4.4] (also cf. (3.8)), and the remaining assertions are [BL85, Propositions 2.2 and 2.3]. \blacksquare

Let \mathfrak{X} be a semistable formal model for \widehat{X} . We let $V(\mathfrak{X})$ denote the inverse image of the set of generic points of $\overline{\mathfrak{X}}$ under the reduction map. This is a finite set of type-2 points of \mathfrak{X}^{an} that maps bijectively onto the set of generic points of $\overline{\mathfrak{X}}$.

Corollary 5.35. *Let \mathfrak{X} be a semistable formal model for \widehat{X} . Then $V(\mathfrak{X})$ is a semistable vertex set of \widehat{X} , and the decomposition of $\widehat{X}^{\text{an}} \setminus V(\mathfrak{X})$ into formal fibers is a semistable decomposition.*

Proof. By [Thu05, Lemme 2.1.13] the formal fibers of \mathfrak{X} are the connected components of $\widehat{X}^{\text{an}} \setminus V(\mathfrak{X})$, so the assertion reduces to Theorem 5.34. \blacksquare

5.36. Let \mathfrak{X} be a semistable formal model for \widehat{X} . Let $\xi \in \overline{\mathfrak{X}}$ be a singular point and let $z_1, z_2 \in \widehat{X}^{\text{an}}$ be the inverse images of the generic points of $\overline{\mathfrak{X}}$ specializing to ξ (it may be that $z_1 = z_2$). Then z_1, z_2 are the vertices of the edge in $\Sigma(\widehat{X}, V(\mathfrak{X}))$ whose interior is $\Sigma(\text{red}^{-1}(\xi))$ by the anti-continuity of the reduction map and Lemma 5.16(2). It follows that $\Sigma(\widehat{X}, V(\mathfrak{X}))$ is the incidence graph of $\overline{\mathfrak{X}}$

(cf. Remark 5.20 and (2.4)). In other words, the vertices of $\Sigma(\widehat{X}, V(\mathfrak{X}))$ correspond to irreducible components of $\overline{\mathfrak{X}}$ and the edges of $\Sigma(\widehat{X}, V(\mathfrak{X}))$ correspond to the points where the components of $\overline{\mathfrak{X}}$ intersect. Moreover, \mathfrak{X} admits an étale map to some $\mathfrak{S}(a) = \mathrm{Spf}(R\langle x, y \rangle / (xy - a))$ in a neighborhood of ξ , and $\mathrm{val}(a)$ is the length of the edge corresponding to ξ (see the proof of Proposition 5.37).

It is clear from the above that a semistable formal model \mathfrak{X} for \widehat{X} is strongly semistable if and only if $V(\mathfrak{X})$ is a strongly semistable vertex set.

Berkovich [Ber04] and Thuillier [Thu05] define the skeleton of a *strongly* semistable formal R -curve using Proposition 5.31. In order to use their results, we must show that the two notions of the skeleton agree:

Proposition 5.37. *Let \mathfrak{X} be a strongly semistable formal model for \widehat{X} . The skeleton $\Sigma(\widehat{X}, V(\mathfrak{X}))$ is naturally identified with the skeleton of \mathfrak{X} defined in [Thu05] as dimension-1 abstract G -affine polyhedral complexes.*

Proof. Thuillier [Thu05, Définition 2.2.13] defines the skeleton $S(\mathfrak{X})$ of \mathfrak{X} to be the set of all points that do not admit an affinoid neighborhood isomorphic to $\mathbf{B}(1)$ and disjoint from $V(\mathfrak{X})$, so $\Sigma(\widehat{X}, V(\mathfrak{X})) = S(\mathfrak{X})$ as sets by Lemma 5.18(4). Let $\xi \in \overline{\mathfrak{X}}$ be a singular point and let \mathfrak{U} be a formal affine neighborhood of ξ admitting an étale morphism $\varphi : \mathfrak{U} \rightarrow \mathfrak{S}(a)$ and inducing an isomorphism $\mathrm{red}^{-1}(\xi) \xrightarrow{\sim} \mathbf{S}(a)_+$ as in Proposition 5.31. Shrinking \mathfrak{U} if necessary, we may and do assume that ξ is the only singular point of $\overline{\mathfrak{U}}$ and that $\overline{\mathfrak{U}}$ has two generic points ζ_1, ζ_2 . Let $z_1, z_2 \in V(\mathfrak{X})$ be the inverse images of ζ_1, ζ_2 . Then $\Sigma(X, V(\mathfrak{X})) \cap \mathfrak{U}^{\mathrm{an}}$ is the edge in $\Sigma(X, V(\mathfrak{X}))$ connecting z_1, z_2 with interior $\Sigma(\mathrm{red}^{-1}(\xi))$. Since φ^{an} maps $\mathrm{red}^{-1}(\xi)$ isomorphically onto $\mathbf{S}(a)_+$ it induces an isometry $\Sigma(X, V(\mathfrak{X})) \cap \mathfrak{U}^{\mathrm{an}} \xrightarrow{\sim} \Sigma(\mathbf{S}(a))$. The polyhedral structure on $S(\mathfrak{X}) \cap \mathfrak{U}^{\mathrm{an}}$ is more or less by definition induced by the identification of $\Sigma(\mathbf{S}(a))$ with $[0, \mathrm{val}(a)]$; see [Thu05, Théorème 2.2.10]. Hence $\Sigma(\widehat{X}, V(\mathfrak{X})) = S(\mathfrak{X})$ as G -affine polyhedral complexes. \blacksquare

In order to prove that semistable vertex sets are in one-to-one correspondence with semistable models as above, it remains to construct a semistable model from a semistable decomposition. The following theorem is folklore; while it is well-known to experts, and in some sense is implicit in [Tem10], we have been unable to find an explicit reference.

Theorem 5.38. *The association $\mathfrak{X} \mapsto V(\mathfrak{X})$ sets up a bijection between the set of semistable formal models of \widehat{X} (up to isomorphism) and the set of semistable vertex sets of \widehat{X} . Furthermore, \mathfrak{X} dominates \mathfrak{X}' if and only if $V(\mathfrak{X}') \subset V(\mathfrak{X})$.*

We will need the following lemmas in the proof of Theorem 5.38.

Lemma 5.39.

- (1) *Let $B \subset \widehat{X}^{\mathrm{an}}$ be an analytic open subset isomorphic to an open ball. Then $\widehat{X}^{\mathrm{an}} \setminus B$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$.*
- (2) *Let $A \subset \widehat{X}^{\mathrm{an}}$ be an analytic open subset isomorphic to an open annulus. Then $\widehat{X}^{\mathrm{an}} \setminus A$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$.*

Proof. First we establish (1). Let us fix an isomorphism $B \cong \mathbf{B}(1)_+$. By [BL85, Lemma 3.5(c)], for any $a \in K^\times$ with $|a| < 1$ the compact set $\widehat{X}^{\mathrm{an}} \setminus \mathbf{B}(a)_+$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$. The limit boundary of $\widehat{X}^{\mathrm{an}} \setminus \mathbf{B}(a)_+$ in $\widehat{X}^{\mathrm{an}}$ is the Gauss point $\|\cdot\|_{\mathrm{val}(a)}$ of $\mathbf{B}(a)$; this coincides with the Shilov boundary of $\widehat{X}^{\mathrm{an}} \setminus \mathbf{B}(a)_+$ by [Thu05, Proposition 2.1.12]. The proof of Lemma 5.16 shows that $\partial_{\lim}(B) = \{x\}$ where $x = \lim_{r \rightarrow 0} \|\cdot\|_r$.

By the Riemann-Roch theorem, there exists a meromorphic function on \widehat{X} which is regular away from $0 \in \mathbf{B}(1)_+$ and which has a zero outside of $\mathbf{B}(1)_+$. Fix such a function f , and scale it so that $|f(x)| = 1$. By Corollary 5.11, the function $F(y) = -\log |f(y)|$ is a monotonically decreasing function on $\Sigma(\mathbf{S}(0)_+) \cong (0, \infty)$ such that $\lim_{r \rightarrow 0} F(\|\cdot\|_r) = 0$. The meromorphic function f defines a finite morphism $\varphi : \widehat{X} \rightarrow \mathbf{P}^1$, which analytifies to a finite morphism $\varphi^{\mathrm{an}} : \widehat{X}^{\mathrm{an}} \rightarrow \mathbf{P}_{\mathrm{an}}^1$. Let $Y = \{y \in \widehat{X}^{\mathrm{an}} : |f(y)| \leq 1\}$ be the inverse image of $\mathbf{B}(1) \subset \mathbf{P}_{\mathrm{an}}^1$ under φ^{an} , so Y is an affinoid

domain in \widehat{X}^{an} . For $a \in \mathfrak{m}_R \setminus \{0\}$ the point $\|\cdot\|_{\text{val}(a)}$ is the Shilov boundary of $\widehat{X}^{\text{an}} \setminus \mathbf{B}(a)_+$, so $|f| \leq \|f\|_{\text{val}(a)}$ on $\widehat{X}^{\text{an}} \setminus \mathbf{B}(a)_+$. Since $\widehat{X}^{\text{an}} \setminus B \subset \widehat{X}^{\text{an}} \setminus \mathbf{B}(a)_+$ for all $a \in \mathfrak{m}_R \setminus \{0\}$ we have $|f| \leq \lim_{r \rightarrow 0} \|f\|_r = 1$ on $\widehat{X}^{\text{an}} \setminus B$. Therefore $\widehat{X}^{\text{an}} \setminus B \subset Y$.

We claim that $\widehat{X}^{\text{an}} \setminus B$ is a connected component of Y . Clearly it is closed in Y . Since f has finitely many zeros in B , there exists $a \in \mathfrak{m}_R \setminus \{0\}$ such that f is a unit on $\mathbf{S}(a)_+ \subset \mathbf{B}(1)_+$. By Lemma 5.13 we have that $|f| > 1$ on $\mathbf{S}(a)_+$, so $\widehat{X}^{\text{an}} \setminus B = (\widehat{X}^{\text{an}} \setminus \mathbf{B}(a)) \cap Y$ is open in Y . Hence $\widehat{X}^{\text{an}} \setminus \mathbf{B}(1)_+$ is affinoid, being a connected component of the affinoid domain Y .

We will reduce the second assertion to the first by doing surgery on \widehat{X}^{an} , following the proof of [Ber93, Proposition 3.6.1]. Let A_1 be a closed annulus inside of A , so $A \setminus A_1 \cong \mathbf{S}(a)_+ \amalg \mathbf{S}(b)_+$ for $a, b \in \mathfrak{m}_R \setminus \{0\}$. Let $(X')^{\text{an}}$ be the analytic curve obtained by gluing $\widehat{X}^{\text{an}} \setminus A_1$ to two copies of $\mathbf{B}(1)_+$ along the inclusions $\mathbf{S}(a)_+ \hookrightarrow \mathbf{B}(1)_+$ and $\mathbf{S}(b)_+ \hookrightarrow \mathbf{B}(1)_+$. One verifies easily that $(X')^{\text{an}}$ is proper in the sense of [Ber90, §3], so $(X')^{\text{an}}$ is the analytification of a unique algebraic curve X' . By construction $\widehat{X}^{\text{an}} \setminus A$ is identified with the affinoid domain $(X')^{\text{an}} \setminus (\mathbf{B}(1)_+ \amalg \mathbf{B}(1)_+)$ in $(X')^{\text{an}}$, so we can apply (1) twice to $(X')^{\text{an}}$ to obtain the result. \blacksquare

Remark 5.40. Let \mathcal{U} be an affinoid domain in \widehat{X}^{an} and let x be a Shilov boundary point of \mathcal{U} . Since $\widetilde{\mathcal{H}}(x)$ is isomorphic to the function field of an irreducible component of the canonical reduction of \mathcal{U} , the point x has type 2. See (3.8) and (3.10). Hence Lemma 5.39 implies that if $A \subset \widehat{X}^{\text{an}}$ is an open ball or an open annulus then $\partial_{\text{lim}}(A)$ consists of either one or two type-2 points of \widehat{X}^{an} .

Recall that if V is a semistable vertex set of \widehat{X} then there is a retraction $\tau_V = \tau_{\Sigma(\widehat{X}, V)} : \widehat{X}^{\text{an}} \rightarrow \Sigma(\widehat{X}, V)$.

Lemma 5.41. *Let V be a semistable vertex set of \widehat{X} and let $x \in V$. Then there are infinitely many open balls in the semistable decomposition for \widehat{X} which retract to x .*

Proof. Suppose that there is at least one edge of $\Sigma(\widehat{X}, V)$. Deleting all of the open annuli in the semistable decomposition of \widehat{X} yields an affinoid domain Y by Lemma 5.39. The set $\tau_V^{-1}(x)$ is a connected component of Y , so $\tau_V^{-1}(x)$ is an affinoid domain as well. The Shilov boundary of $\tau_V^{-1}(x)$ agrees with its limit boundary $\{x\}$ in \widehat{X}^{an} ; by construction $\tau_V^{-1}(x) \setminus \{x\}$ is a disjoint union of open balls, which are the formal fibers of the canonical model of $\tau_V^{-1}(x)$ by [Thu05, Lemme 2.1.13]. Any nonempty curve over k has infinitely many points, so $\tau_V^{-1}(x) \setminus \{x\}$ is a disjoint union of infinitely many open balls.

If $\Sigma(\widehat{X}, V)$ has no edges then $\widehat{X}^{\text{an}} \setminus \{x\}$ is a disjoint union of open balls. Deleting one of these balls yields an affinoid domain by Lemma 5.39, and the above argument goes through. \blacksquare

5.42. Proof of Theorem 5.38. First we prove that $\mathfrak{X} \mapsto V(\mathfrak{X})$ is surjective, i.e., that any semistable vertex set comes from a semistable formal model. Let V be a semistable vertex set of \widehat{X} , let $\Sigma = \Sigma(\widehat{X}, V)$, and let $\tau = \tau_{\Sigma} : \widehat{X}^{\text{an}} \rightarrow \Sigma$ be the retraction.

5.42.1. Case 1. Suppose that Σ has at least two edges. Let e be an edge in Σ , let A_0, A_1, \dots, A_r ($r \geq 1$) be the open annuli in the semistable decomposition of \widehat{X} , and suppose that $\Sigma(A_0)$ is the interior of e . Then $\widehat{X} \setminus (\bigcup_{i=1}^r A_i)$ is an affinoid domain by Lemma 5.39, and $\tau^{-1}(e)$ is a connected component of $\widehat{X} \setminus (\bigcup_{i=1}^r A_i)$. Hence $\tau^{-1}(e)$ is an affinoid domain in \widehat{X}^{an} . Let \mathfrak{Y} be its canonical model. Let $x, y \in \widehat{X}^{\text{an}}$ be the endpoints of e , so $\{x, y\} = \partial_{\text{lim}}(\tau^{-1}(e))$ is the Shilov boundary of $\tau^{-1}(e)$, and $\tau^{-1}(e) \setminus \{x, y\}$ is a disjoint union of open balls and the open annulus A_0 . By [Thu05, Lemme 2.1.13], the formal fibers of $\tau^{-1}(e) \rightarrow \overline{\mathfrak{Y}}$ are the connected components of $\tau^{-1}(e) \setminus \{x, y\}$, so $\overline{\mathfrak{Y}}$ has either one or two irreducible components (depending on whether $x = y$) which intersect along a single ordinary double point ξ by Theorem 5.34. Let \mathfrak{C}_x (resp. \mathfrak{C}_y) be the irreducible component of \mathfrak{Y} whose generic point is the reduction of x (resp. y). Using the anti-continuity of the reduction map one sees that $\text{red}^{-1}(\overline{\mathfrak{C}}_x \setminus \{\xi\}) = \tau^{-1}(x)$ and $\text{red}^{-1}(\overline{\mathfrak{C}}_y \setminus \{\xi\}) = \tau^{-1}(y)$. It follows that the formal

affine subset $\mathfrak{C}_x \setminus \{\xi\}$ (resp. $\mathfrak{C}_y \setminus \{\xi\}$) is the canonical model of the affinoid domain $\tau^{-1}(x)$ (resp. $\tau^{-1}(y)$).

Applying the above for every edge e of Σ allows us to glue the canonical models of the affinoid domains $\tau^{-1}(e)$ together along the canonical models of the affinoid domains $\tau^{-1}(x)$ corresponding to the vertices x of Σ . Thus we obtain a semistable formal model \mathfrak{X} of \widehat{X} such that $V(\mathfrak{X}) = V$ (cf. Remark 5.30(1)).

5.42.2. Case 2. Suppose that Σ has one edge e and two vertices x, y . Let B_x, B'_x (resp. B_y, B'_y) be distinct open balls in the semistable decomposition of \widehat{X} retracting to x (resp. y), so $Y := \widehat{X}^{\text{an}} \setminus (B_x \cup B_y)$ and $Y' := \widehat{X}^{\text{an}} \setminus (B'_x \cup B'_y)$ are affinoid domains by Lemma 5.39. Let \mathfrak{Y} (resp. \mathfrak{Y}') be the canonical model of Y (resp. Y'). Arguing as in Case 1 above, $\overline{\mathfrak{Y}}$ and $\overline{\mathfrak{Y}'}$ are affine curves with two irreducible components intersecting along a single ordinary double point ξ . Furthermore, $Z = Y \cap Y'$ is an affinoid domain whose canonical model \mathfrak{Z} is obtained from \mathfrak{Y} (resp. \mathfrak{Y}') by deleting one smooth point from each component. Gluing \mathfrak{Y} to \mathfrak{Y}' along \mathfrak{Z} yields the desired semistable formal model \mathfrak{X} of \widehat{X} .

5.42.3. Case 3. Suppose that Σ has just one vertex x . Let B, B' be distinct open balls in the semistable decomposition of \widehat{X} , let $Y = \widehat{X}^{\text{an}} \setminus B$, let $Y' = \widehat{X}^{\text{an}} \setminus B'$, and let $Z = Y \cap Y'$. Gluing the canonical models of Y and Y' along the canonical model of Z gives us our semistable formal model as in Case 2.

5.42.4. A semistable formal model of \widehat{X} is determined by its formal fibers [BL85, Lemma 3.10], so $\mathfrak{X} \mapsto V(\mathfrak{X})$ is bijective. It remains to prove that \mathfrak{X} dominates \mathfrak{X}' if and only if $V(\mathfrak{X}') \subset V(\mathfrak{X})$. If \mathfrak{X} dominates \mathfrak{X}' then $V(\mathfrak{X}') \subset V(\mathfrak{X})$ by the surjectivity and functoriality of the reduction map. Conversely let V, V' be semistable vertex sets of \widehat{X} such that $V' \subset V$. The corresponding semistable formal models $\mathfrak{X}, \mathfrak{X}'$ were constructed above by finding coverings $\mathcal{U}, \mathcal{U}'$ of \widehat{X}^{an} by affinoid domains whose canonical models glue along the canonical models of their intersections. (Such a covering is called a *formal covering* in [BL85].) It is clear that if \mathcal{U} refines \mathcal{U}' , in the sense that every affinoid in \mathcal{U} is contained in an affinoid in \mathcal{U}' , then we obtain a morphism $\mathfrak{X} \rightarrow \mathfrak{X}'$ of semistable formal models. Therefore it suffices to show that we can choose $\mathcal{U}, \mathcal{U}'$ such that \mathcal{U} refines \mathcal{U}' when $V' \subset V$ in all of the cases treated above. We will carry out this procedure in the situation of Case 1, when V is the union of V' with a type-2 point $x \in \Sigma' = \Sigma(\widehat{X}, V')$ not contained in V' ; the other cases are similar and are left to the reader (cf. the proof of Proposition 5.27).

In the situation of Case 1, the formal covering corresponding to V' is the set

$$\mathcal{U}' = \{\tau^{-1}(e) : e \text{ is an edge of } \Sigma'\}.$$

By Proposition 5.27(2) the skeleton $\Sigma = \Sigma(\widehat{X}, V)$ is a refinement of Σ' , obtained by subdividing the edge e_0 containing x to allow x as a vertex. Let e_1, e_2 be the edges of Σ containing x . Then $\tau^{-1}(e_1), \tau^{-1}(e_2)$ are affinoid domains in \widehat{X}^{an} contained in $\tau^{-1}(e_0)$, so the formal covering $\mathcal{U} = \{\tau^{-1}(e) : e \text{ is an edge of } \Sigma\}$ is a refinement of \mathcal{U}' , as desired. \blacksquare

5.43. Stable models and the minimal skeleton. Here we explain when and in what sense there exists a minimal semistable vertex set of X . Of course this question essentially reduces to the existence of a stable model of X when $X = \widehat{X}$; using [BL85] we can also treat the case when X is not proper.

Definition. Let $x \in X^{\text{an}}$ be a type-2 point. The *genus* of x , denoted $g(x)$, is defined to be the genus of the smooth proper connected k -curve with function field $\widetilde{\mathcal{H}}(x)$.

Remark 5.44. Let V be a semistable vertex set of \widehat{X} and let $x \in \widehat{X}^{\text{an}}$ be a type-2 point with positive genus. Then $x \in V$, since otherwise x admits a neighborhood which is isomorphic to an analytic domain in \mathbf{P}_{an}^1 and the genus of any type-2 point in \mathbf{P}_{an}^1 is zero.

Remark 5.45. Let \mathfrak{X} be a semistable formal model for \widehat{X} , let $x \in V(\mathfrak{X})$, and let $\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}$ be the irreducible component with generic point $\zeta = \text{red}(x)$. Then $\widetilde{\mathcal{H}}(x)$ is isomorphic to $\mathcal{O}_{\overline{\mathfrak{X}}, \zeta}$ by [Ber90,

Proposition 2.4.4], so $g(x)$ is the genus of the normalization of $\bar{\mathfrak{C}}$. It follows from [BL85, Theorem 4.6] that

$$(5.45.1) \quad g(\widehat{X}) = \sum_{x \in V(\mathfrak{X})} g(x) + g(\Sigma(\widehat{X}, V))$$

where $g(\widehat{X})$ is the genus of \widehat{X} and $g(\Sigma(\widehat{X}, V)) = \text{rank}_{\mathbf{Z}}(H_1(\Sigma(\widehat{X}, V), \mathbf{Z}))$ is the genus of $\Sigma(\widehat{X}, V)$ as a topological space (otherwise known as the cyclomatic number of the graph $\Sigma(\widehat{X}, V)$). The important equation (5.45.1) is known as the *genus formula*.

Definition 5.46. The *Euler characteristic* of X is defined to be

$$\chi(X) = 2 - 2g(\widehat{X}) - \#D.$$

Definition 5.47. A semistable vertex set V of X is *stable* if there is no $x \in V$ of genus zero and valence less than three in $\Sigma(X, V)$. We call the corresponding semistable decomposition of X *stable* as well. A semistable formal model \mathfrak{X} of \widehat{X} such that $V(\mathfrak{X})$ is a stable vertex set of \widehat{X} is called a *stable formal model*.

A semistable vertex set V of X is *minimal* if V does not properly contain a semistable vertex set V' . Any semistable vertex set contains a minimal one.

Proposition 5.48. Let V be a semistable vertex set of X and let $x \in V$ be a point of genus zero.

- (1) Suppose that x has valence one in $\widehat{\Sigma}(X, V)$, let e be the edge adjoining x , and let y be the other endpoint of e . If $y \notin D$ then $V \setminus \{x\}$ is a semistable vertex set of X and $\widehat{\Sigma}(X, V \setminus \{x\})$ is the graph obtained from $\widehat{\Sigma}(X, V)$ by removing x and the interior of e .
- (2) Suppose that x has valence two in $\widehat{\Sigma}(X, V)$, let e_1, e_2 be the edges adjoining x , and let x_1 (resp. x_2) be the other endpoint of e_1 (resp. e_2). If $\{x_1, x_2\} \not\subset D$ then $V \setminus \{x\}$ is a semistable vertex set of X and $\widehat{\Sigma}(X, V \setminus \{x\})$ is the graph obtained from $\widehat{\Sigma}(X, V)$ by joining e_1, e_2 into a single edge.

Proof. This is essentially [BL85, Lemma 6.1] translated into our language. ■

By a *topological vertex* of a finite connected graph Γ we mean a vertex of valence at least 3. The set of topological vertices only depends on the topological realization of Γ .

Theorem 5.49. (Stable reduction theorem) *There exists a semistable vertex set of X . If V is a minimal semistable vertex set of X then:*

- (1) If $\chi(X) \leq 0$ then $\Sigma(X, V)$ is the set of points in X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbf{B}(1)$.
- (2) If $\chi(X) < 0$ then V is stable and

$$V = \{x \in \Sigma(X, V) : x \text{ is a topological vertex of } \Sigma(X, V) \text{ or } g(x) > 0\}.$$

Corollary 5.50. *If $\chi(X) \leq 0$ then there is a unique set-theoretic minimal skeleton of X , and if $\chi(X) < 0$ then there is a unique stable vertex set of X .*

Proof of Theorem 5.49. The existence of a semistable vertex set of \widehat{X} follows from the classic theorem of Deligne and Mumford [DM69] as proved analytically (over a non-noetherian rank-1 valuation ring) in [BL85, Theorem 7.1]. The existence of a semistable vertex set of X then follows from Proposition 5.27(3). Let V be a minimal semistable vertex set of X and let $\Sigma = \Sigma(X, V)$. If $\chi(X) < 0$ then one applies Proposition 5.48 in the standard way to prove the second assertion, and if $\chi(X) \leq 0$ then Proposition 5.48(1) guarantees that every genus-zero vertex of Σ has valence at least two.

Suppose that $\chi(X) \leq 0$. Let Σ' be the set of points of X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbf{B}(1)$. By Lemma 5.18(4) we have $\Sigma' \subset \Sigma$. Let $x \in \Sigma$, and suppose that x admits an affinoid neighborhood U isomorphic to $\mathbf{B}(1)$. We will show by way of contradiction that Σ has a vertex of valence less than two in U (any vertex contained in U has genus zero); in fact we will show

that $\Sigma \cap U$ is a tree. Let y be the Gauss point of U . If $y \in \Sigma$ then we may replace V by $V \cup \{y\}$ by Proposition 5.27(2) to assume that $y \in V$. Since U is closed and any connected component of $X^{\text{an}} \setminus V$ that intersects U is contained in U , the retraction $\tau_\Sigma : X^{\text{an}} \rightarrow \Sigma$ restricts to a retraction $U \rightarrow U \cap \Sigma$. Since U is contractible, $U \cap \Sigma$ is a tree as claimed. \blacksquare

Remark 5.51. If $\chi(X) = 0$ then either $g(X) = 0$ and $\#D = 2$ or $g(X) = 1$ and $\#D = 0$. In the first case, the skeleton of $X \cong \mathbf{G}_m$ is the line connecting 0 and ∞ , and any type-2 point on this line is a minimal semistable vertex set. In the second case, $X = \widehat{X}$ is an elliptic curve with respect to some choice of distinguished point $0 \in X(K)$. If X has good reduction then there is a unique point $x \in X^{\text{an}}$ with $g(x) = 1$; in this case $\{x\}$ is the unique stable vertex set of X and $\Sigma(X, \{x\}) = \{x\}$.

Suppose now that $(X, 0)$ is an elliptic curve with multiplicative reduction, i.e., X is a Tate curve. By Tate's uniformization theory [BGR84, §9.7], there is a unique $q = q_X \in K^\times$ with $\text{val}(q) > 0$ and an étale morphism $u : \mathbf{G}_m^{\text{an}} \rightarrow X^{\text{an}}$ which is a homomorphism of group objects (in the category of K -analytic spaces) with kernel $u^{-1}(0) = q^{\mathbf{Z}}$. For brevity we will often write $X^{\text{an}} \cong \mathbf{G}_m^{\text{an}}/q^{\mathbf{Z}}$. The so-called Tate parameter q is related to the j -invariant $j = j_X$ of X in such a way that $\text{val}(q) = -\text{val}(j)$. Let Z be the retraction of the set $q^{\mathbf{Z}}$ onto the skeleton of \mathbf{G}_m , i.e., the collection of Gauss points of the balls $\mathbf{B}(q^n)$ for $n \in \mathbf{Z}$. Then $\mathbf{G}_m^{\text{an}} \setminus Z$ is the disjoint union of the open annuli $\{\mathbf{S}(q^{n+1}, q^n)_+\}_{n \in \mathbf{Z}}$ and infinitely many open balls, and every connected component of $\mathbf{G}_m^{\text{an}} \setminus Z$ maps isomorphically onto its image in X^{an} . It follows that $X^{\text{an}} \setminus \{u(1)\}$ is a disjoint union of an open annulus A isomorphic to $\mathbf{S}(q)_+$ and infinitely many open balls. Hence $V = \{u(1)\}$ is a (minimal) semistable vertex set of X , and the associated (minimal) skeleton Σ is a circle of circumference $\text{val}(q) = -\text{val}(j_E)$. We have $u(1) = \tau_\Sigma(0)$, so any type-2 point on Σ is a minimal semistable vertex set, as any such point is the retraction of a K -point of X (which we could have chosen to be 0).

See also [Gub03, Example 7.20].

Remark 5.52. Given a smooth complete curve \widehat{X}/K of genus g and a subset D of ‘marked points’ of $\widehat{X}(K)$ satisfying the inequality $2 - 2g - n \leq 0$, where $n = \#D \geq 0$, one obtains a canonical pair (Γ, w) consisting of an abstract metric graph and a vertex weight function, where $\Gamma = \Sigma(\widehat{X} \setminus D, V)$ is the minimal skeleton of $\widehat{X} \setminus D$ and $w : \Gamma \rightarrow \mathbf{Z}_{\geq 0}$ takes $x \in \Gamma$ to 0 if $x \notin V$ and to $g(x)$ if $x \in V$. (A closely related construction can be found in [Tyo10, §2].) If $2 - 2g - n < 0$, this gives a canonical ‘abstract tropicalization map’ $\text{trop} : M_{g,n} \rightarrow M_{g,n}^{\text{trop}}$, where $M_{g,n}^{\text{trop}}$ is the moduli space of n -pointed tropical curves of genus g as defined, for example, in [Cap11, §3]. The map $\text{trop} : M_{g,n} \rightarrow M_{g,n}^{\text{trop}}$ is certainly deserving of further study.

Corollary 5.53. *Let $x \in X^{\text{an}}$. There is a fundamental system of open neighborhoods $\{U_\alpha\}$ of x of the following form:*

- (1) *If x is a type-1 or a type-4 point then the U_α are open balls.*
- (2) *If x is a type-3 point then the U_α are open annuli with $x \in \Sigma(U_\alpha)$.*
- (3) *If x is a type-2 point then $U_\alpha = \tau_V^{-1}(W_\alpha)$ where W_α is a simply-connected open neighborhood of x in $\Sigma(X, V)$ for some semistable vertex set V of X containing x , and each $U_\alpha \setminus \{x\}$ is a disjoint union of open balls and open annuli.*

Proof. Since X has a semistable decomposition, if x is a point of type 1, 3, or 4 then x has a neighborhood isomorphic to an open annulus or an open ball. Hence we may assume that $X = \mathbf{P}^1$ and $x \in \mathbf{B}(1)_+$. By [BR10, Proposition 1.6] the set of open balls with finitely many closed balls removed forms a basis for the topology on $\mathbf{B}(1)_+$; assertions (1) and (2) follow easily from this.

Let f be a meromorphic function on X ; deleting the zeros and poles of f , we may assume that f is a unit on X . Let $F = \log |f| : X^{\text{an}} \rightarrow \mathbf{R}$ and let $U = F^{-1}((a, b))$ for some interval $(a, b) \subset \mathbf{R}$. Let x be a type-2 point contained in U . Since such U form a sub-basis for the topology on X^{an} it suffices to prove that there is a neighborhood of x of the form described in (3) contained in U . Let V be a semistable vertex set for X containing x . By Proposition 5.5 and Lemma 5.13 we have that F is linear on the edges of $\Sigma(X, V)$ and that F factors through $\tau_V : X^{\text{an}} \rightarrow \Sigma(X, V)$. Therefore if W is any simply-connected neighborhood of x in $\Sigma(X, V)$ contained in $U = F^{-1}((a, b))$ then $\tau_V^{-1}(W) \subset U$.

If we assume in addition that the intersection of W with any edge of Σ adjoining x is a half-open interval with endpoints in G then $\tau_V^{-1}(W) \setminus \{x\}$ is a disjoint union of open balls and open annuli. ■

Definition 5.54. A neighborhood of $x \in X^{\text{an}}$ of the form described in Corollary 5.53 is called a *simple neighborhood* of x .

5.55. A simple neighborhood of a type-2 point $x \in X^{\text{an}}$ has the following alternative description. Let V be a semistable vertex set containing x and let W be a simply-connected neighborhood of x in $\Sigma(X, V)$ such that the intersection of W with any edge adjoining x is a half-open interval with endpoints in G , so $U = \tau_V^{-1}(W)$ is a simple neighborhood of x . Adding the boundary of W to V , we may assume that the connected components of $U \setminus \{x\}$ are connected components of $X^{\text{an}} \setminus V$. Let \mathfrak{X} be the semistable formal model of \widehat{X} associated to V and let $\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}$ be the irreducible component with generic point $\text{red}(x)$. Since W contains no loop edges of $\Sigma(X, V)$, the component $\overline{\mathfrak{C}}$ is smooth. The connected components of $\widehat{X}^{\text{an}} \setminus V$ are the formal fibers of \mathfrak{X} , so it follows from the anti-continuity of red that $U = \text{red}^{-1}(\overline{\mathfrak{C}})$ and that $\pi_0(U \setminus \{x\}) \xrightarrow{\sim} \overline{\mathfrak{C}}(k)$. To summarize:

Lemma. A simple neighborhood U of a type-2 point $x \in X^{\text{an}}$ is the inverse image of a smooth irreducible component $\overline{\mathfrak{C}}$ of the special fiber of a semistable formal model \mathfrak{X} of \widehat{X} . Furthermore, we have $\pi_0(U \setminus \{x\}) \xrightarrow{\sim} \overline{\mathfrak{C}}(k)$.

The set of all skeleta $\{\Sigma(X, V)\}_V$ is a filtered directed system under inclusion by Proposition 5.27. Recall from (3.10) that the set of *skeletal points* $\mathbf{H}_o(X^{\text{an}})$ of X^{an} is by definition the set of points of X^{an} of types 2 and 3, and the collection of *norm points* $\mathbf{H}(X^{\text{an}})$ is $X^{\text{an}} \setminus X(K)$. The following corollary explains the former terminology:

Corollary 5.56. We have

$$\mathbf{H}_o(X^{\text{an}}) = \bigcup_V \Sigma(X, V) = \varinjlim_V \Sigma(X, V)$$

as sets, where V runs over all semistable vertex sets of X .

Proof. Any point of $\Sigma(X, V)$ has type 2 or 3, and any type-2 point is contained in a semistable vertex set by Proposition 5.27(3). Let x be a type-3 point. Then x is contained in an open ball or an open annulus in a semistable decomposition of X^{an} . The semistable decomposition can then be refined as in the proof of Proposition 5.27(3) to produce a skeleton that includes x . ■

By Proposition 5.27(1), the set of all skeleta $\{\Sigma(X, V)\}_V$ is also an inverse system with respect to the natural retraction maps. Although not logically necessary for anything else in this paper, the following folklore counterpart to Corollary 5.56 is conceptually important. For a higher-dimensional analogue (without proof) in the case $\text{char}(K) = 0$, see [KS06, Appendix A].

Theorem 5.57. The natural map

$$u : \widehat{X}^{\text{an}} \rightarrow \varprojlim_V \Sigma(\widehat{X}, V)$$

is a homeomorphism of topological spaces, where V runs over all semistable vertex sets of \widehat{X} .

Proof. The map u exists and is continuous by the universal property of inverse limits. It is injective because given any two points $x \neq y$ in \widehat{X}^{an} , one sees easily that there is a semistable vertex set V such that x and y retract to different points of $\Sigma(\widehat{X}, V)$. Since \widehat{X}^{an} is compact and each individual retraction map $\widehat{X}^{\text{an}} \rightarrow \Sigma(\widehat{X}, V)$ is continuous and surjective, it follows from [Bou98, §9.6 Corollary 2] that u is also surjective. By Proposition 8 in §9.6 of loc. cit., the space $\varprojlim_V \Sigma(\widehat{X}, V)$ is compact. Therefore u is a continuous bijection between compact (Hausdorff) spaces, hence a homeomorphism (cf. Corollary 2 in §9.4 of loc. cit.). ■

5.58. The metric structure on $\mathbf{H}_o(X^{\text{an}})$. Let $V \subset V'$ be semistable vertex sets of X . By Proposition 5.27(3) every edge e of $\Sigma(X, V)$ includes isometrically into an edge of $\Sigma(X, V')$. Let $x, y \in \Sigma(X, V)$ and let $[x, y]$ be a shortest path from x to y in $\Sigma(X, V)$. Then $[x, y]$ is also a shortest path in $\Sigma(X, V')$: if there were a shorter path $[x, y]'$ in $\Sigma(X, V')$ then $[x, y] \cup [x, y]'$ would represent a

homology class in $H_1(\Sigma(X, V'), \mathbf{Z})$ that did not exist in $H_1(\Sigma(X, V), \mathbf{Z})$, which is impossible by the genus formula (5.45.1). Therefore the inclusion $\Sigma(X, V) \hookrightarrow \Sigma(X, V')$ is an isometry (with respect to the shortest-path metrics), so by Corollary 5.56 we obtain a natural metric ρ on $\mathbf{H}_o(X^{\text{an}})$, called the *skeletal metric*.

Let V be a semistable vertex set and let $\tau = \tau_V : X^{\text{an}} \rightarrow \Sigma(X, V)$ be the retraction onto the skeleton. If $x, y \in \mathbf{H}_o(X^{\text{an}})$ are not contained in the same connected component of $X^{\text{an}} \setminus \Sigma(X, V)$ then a shortest path from x to y in a larger skeleton must go through $\Sigma(X, V)$. It follows that

$$(5.58.1) \quad \rho(x, y) = \rho(x, \tau(x)) + \rho(\tau(x), \tau(y)) + \rho(\tau(y), y).$$

Remark 5.59.

- (1) By definition any skeleton includes isometrically into $\mathbf{H}_o(X^{\text{an}})$.
- (2) It is important to note that the metric topology on $\mathbf{H}_o(X^{\text{an}})$ is *stronger* than the subspace topology.

We can describe the skeletal metric locally as follows. By Berkovich's classification theorem, any point $x \in \mathbf{H}(\mathbf{A}_{\text{an}}^1)$ is a limit of Gauss points of balls of radii r_i converging to $r \in (0, \infty)$. We define $\text{diam}(x) = r$. Any two points $x \neq y \in \mathbf{A}_{\text{an}}^1$ are contained in a unique smallest closed ball; its Gauss point is denoted $x \vee y$. For $x, y \in \mathbf{H}(\mathbf{A}_{\text{an}}^1)$ we define

$$\rho_p(x, y) = 2 \log(\text{diam}(x \vee y)) - \log(\text{diam}(x)) - \log(\text{diam}(y)).$$

Then ρ_p is a metric on $\mathbf{H}(\mathbf{A}_{\text{an}}^1)$, called the *path distance metric*; see (2.4) and [BR10, §2.7]. If A is a standard open ball or standard generalized open annulus then the restriction of ρ_p to $\mathbf{H}(A)$ is called the *path distance metric* on $\mathbf{H}(A)$.

Proposition 5.60. *Let $A \subset X^{\text{an}}$ be an analytic domain isomorphic to a standard open ball or a standard generalized open annulus. Then the skeletal metric on $\mathbf{H}_o(X^{\text{an}})$ and the path distance metric on $\mathbf{H}(A)$ restrict to the same metric on $\mathbf{H}_o(A)$.*

Proof. Let V be a semistable vertex set containing the limit boundary of A (cf. Remark 5.40). Then $V \setminus (V \cap A)$ is a semistable vertex set since the connected components of $A \setminus (V \cap A)$ are connected components of $X^{\text{an}} \setminus V$. Hence we may and do assume that A is a connected component of $X^{\text{an}} \setminus V$. Suppose that A is an open ball, and fix an isomorphism $A \cong \mathbf{B}(a)_+$. Let $x, y \in A$ be type-2 points.

- (1) Suppose that $x \vee y \in \{x, y\}$; without loss of generality we may assume that $x = x \vee y$. After recentering, we may assume in addition that x is the Gauss point of $\mathbf{B}(b)$ and that y is the Gauss point of $\mathbf{B}(c)$. Then the standard open annulus $A' = \mathbf{B}(b)_+ \setminus \mathbf{B}(c)$ is a connected component of $A \setminus \{x, y\}$, which breaks up into a disjoint union of open balls and the open annuli A' and $\mathbf{B}(a)_+ \setminus \mathbf{B}(b)$. Hence $V \cup \{x, y\}$ is a semistable vertex set, and $\Sigma(A')$ is the interior of the edge e of $\Sigma(X, V \cup \{x, y\})$ with endpoints x, y . Therefore $\rho(x, y)$ is the logarithmic modulus of A' , which agrees with $\rho_p(x, y) = \log(\text{diam}(x)) - \log(\text{diam}(y))$.
- (2) Suppose that $z = x \vee y \notin \{x, y\}$. Then $A \setminus \{x, y, z\}$ is a disjoint union of open balls and three open annuli, two of which connect x, z and y, z . As above we have $\rho(x, y) = \rho(x, z) + \rho(y, z)$, which is the same as $\rho_p(x, y) = (\log(\text{diam}(z)) - \log(\text{diam}(x))) + (\log(\text{diam}(z)) - \log(\text{diam}(y)))$.

Since the type-2 points of A are dense [BR10, Lemma 1.8], this proves the claim when A is an open ball in a semistable decomposition of X . The proof when A is a generalized open annulus in a semistable decomposition of X has more cases but is not essentially any different, so it is left to the reader. \blacksquare

Since Proposition 5.60 did not depend on the choice of isomorphism of A with a standard generalized open annulus, we obtain:

Corollary 5.61. *Any isomorphism of standard open balls or standard generalized open annuli induces an isometry with respect to the path distance metric.*

In particular, if A is an (abstract) open ball or generalized open annulus then we can speak of the path distance metric on $\mathbf{H}(A)$.

Corollary 5.62. *The metric ρ on $\mathbf{H}_o(X^{\text{an}})$ extends in a unique way to a metric on $\mathbf{H}(X^{\text{an}})$.*

Proof. Let $x, y \in \mathbf{H}(X^{\text{an}})$ and let V be a semistable vertex set of X . If x, y are contained in the same connected component $B \cong \mathbf{B}(1)_+$ of $X^{\text{an}} \setminus \Sigma(X, V)$ then we set $\rho(x, y) = \rho_p(x, y)$. Otherwise we set

$$\rho(x, y) = \rho_p(x, \tau_V(x)) + \rho(\tau_V(x), \tau_V(y)) + \rho_p(\tau_V(y), y)$$

where we have extended the path distance metric ρ_p on a connected component B of $X^{\text{an}} \setminus \Sigma(X)$ to its closure $B \cup \tau_V(B)$ by continuity (compare the proof of Lemma 5.16). By (5.58.1) and Proposition 5.60 this function extends ρ . We leave it to the reader to verify that ρ is a metric on $\mathbf{H}(X^{\text{an}})$. ■

A *geodesic segment* from x to y in a metric space T is the image of an isometric embedding $[a, b] \hookrightarrow T$ with $a \mapsto x$ and $b \mapsto y$. We often identify a geodesic segment with its image in T . Recall that an **R-tree** is a metric space T with the following properties:

- (1) For all $x, y \in T$ there is a unique geodesic segment $[x, y]$ from x to y .
- (2) For all $x, y, z \in T$, if $[x, y] \cap [y, z] = \{y\}$ then $[x, z] = [x, y] \cup [y, z]$.

See [BR10, Appendix B]. It is proved in §1.4 of loc. cit. that $\mathbf{H}(\mathbf{B}(1))$ is an **R-tree** under the path distance metric. It is clear that any path-connected subspace of an **R-tree** is an **R-tree**, so if A is an open ball or a generalized open annulus then $\mathbf{H}(A)$ is an **R-tree** as well.

Proposition 5.63. *Every point $x \in \mathbf{H}(X^{\text{an}})$ admits a fundamental system of simple neighborhoods $\{U_\alpha\}$ in X^{an} such that $U_\alpha \cap \mathbf{H}(X^{\text{an}})$ is an **R-tree** under the restriction of ρ .*

The definition of a simple neighborhood of a point $x \in X^{\text{an}}$ is found in (5.54).

Proof. If x has type 3 or 4 then a simple neighborhood of x is an open ball or an open annulus, so the proposition follows from Corollary 5.53 and Proposition 5.60. Let x be a type-2 point and let V be a semistable vertex set of X containing x . For small enough $\varepsilon > 0$ the set $W = \{y \in \Sigma(X, V) : \rho(x, y) < \varepsilon\}$ is simply-connected; fix such an $\varepsilon \in G$, and let $U = \tau_V^{-1}(W)$. Then U is a simple neighborhood of x . We claim that $\mathbf{H}(U)$ is an **R-tree**. Any connected component A of $U \setminus \{x\}$ is an open ball or an open annulus, so $\mathbf{H}(A)$ is an **R-tree**. Moreover $\mathbf{H}(A) \cup \{x\}$ is isometric to a path-connected subspace of $\mathbf{H}(\mathbf{B}(1))$ as in the proof of Lemma 5.16; it follows that $\mathbf{H}(A) \cup \{x\}$ is an **R-tree**. Therefore $\mathbf{H}(U)$ is a collection of **R-trees** joined together at the single point x , and the hypotheses on W along with (5.58.1) imply that if $y, z \in \mathbf{H}(U)$ are contained in different components of $U \setminus \{x\}$ then $\rho(y, z) = \rho(y, x) + \rho(x, z)$. It is clear that such an object is again an **R-tree**. ■

Corollary 5.64. *Let $x, y \in \mathbf{H}_o(X^{\text{an}})$ and let $\Sigma = \Sigma(X, V)$ be a skeleton containing x and y . Then any geodesic segment from x to y is contained in Σ .*

Proof. Any path from x to y in Σ is by definition a geodesic segment. If x, y are contained in an open subset U such that $\mathbf{H}(U)$ is an **R-tree** then the path from x to y in $\Sigma \cap U$ is the unique geodesic segment from x to y in $\mathbf{H}(U)$. The general case follows by covering a geodesic segment from x to y by (finitely many) such U . ■

5.65. Tangent directions and the Slope Formula. Let $x \in \mathbf{H}(X^{\text{an}})$. A *nontrivial geodesic segment starting at x* is a geodesic segment $\alpha : [0, a] \hookrightarrow \mathbf{H}(X^{\text{an}})$ with $a > 0$ such that $\alpha(0) = x$. We say that two nontrivial geodesic segments α, α' starting at x are *equivalent at x* if α and α' agree on a neighborhood of 0. Following [BR10, §B.6], we define the set of *tangent directions at x* to be the set T_x of nontrivial geodesic segments starting at x up to equivalence at x . It is clear that T_x only depends on a neighborhood of x in X^{an} .

Lemma 5.66. *Let $x \in \mathbf{H}(X^{\text{an}})$ and let U be a simple neighborhood of x in X^{an} . Then $[x, y] \mapsto y$ establishes a bijection $T_x \xrightarrow{\sim} \pi_0(U \setminus \{x\})$. Moreover,*

- (1) *If x has type 4 then there is only one tangent direction at x .*
- (2) *If x has type 3 then there are two tangent directions at x .*
- (3) *If x has type 2 then $U = \text{red}^{-1}(\overline{\mathcal{C}})$ for a smooth irreducible component $\overline{\mathcal{C}}$ of the special fiber of a semistable formal model \mathfrak{X} of \widehat{X} by (5.55), and $T_x \xrightarrow{\sim} \pi_0(U \setminus \{x\}) \xrightarrow{\sim} \overline{\mathcal{C}}(k)$.*

Proof. We will assume for simplicity that $\mathbf{H}(U)$ is an \mathbf{R} -tree (i.e., that the induced metric on $\mathbf{H}(U)$ agrees with the shortest-path metric); the general case reduces to this because U is contractible. The bijection $T_x \xrightarrow{\sim} \pi_0(\mathbf{H}(U) \setminus \{x\})$ is proved in [BR10, §B.6]. A connected component B of $U \setminus \{x\}$ is an \mathbf{R} -tree by Proposition 1.13 of loc. cit. and the type-1 points of B are leaves, so $\pi_0(\mathbf{H}(U) \setminus \{x\}) = \pi_0(U \setminus \{x\})$. Parts (1) and (2) are proved in §1.4 of loc. cit. and part (3) is (5.55). ■

5.67. With the notation in Lemma 5.66(3), we have a canonical identification of $\widetilde{\mathcal{H}}(x)$ with the function field of $\overline{\mathcal{C}}$ by [Ber90, Proposition 2.4.4]. Hence we have an identification $\xi \mapsto \text{ord}_\xi$ of $\overline{\mathcal{C}}(k)$ with the set $\text{DV}(\widetilde{\mathcal{H}}(x)/k)$ of nontrivial discrete valuations $\widetilde{\mathcal{H}}(x) \rightarrow \mathbf{Z}$ inducing the trivial valuation on k . One can prove that the composite bijection $T_x \xrightarrow{\sim} \text{DV}(\widetilde{\mathcal{H}}(x)/k)$ is independent of the choice of U . The discrete valuation corresponding to a tangent direction $v \in T_x$ will be denoted $\text{ord}_v : \widetilde{\mathcal{H}}(x) \rightarrow \mathbf{Z}$.

Let $x \in X^{\text{an}}$ be a type-2 point and let f be an analytic function in a neighborhood of x . Let $c \in K^\times$ be a scalar such that $|f(x)| = c$. We define $\tilde{f}_x \in \widetilde{\mathcal{H}}(x)$ to be the residue of $c^{-1}f$, so \tilde{f}_x is only defined up to multiplication by a nonzero scalar in k . However if $\text{ord} : \widetilde{\mathcal{H}}(x) \rightarrow \mathbf{Z}$ is a nontrivial discrete valuation trivial on k then $\text{ord}(\tilde{f}_x)$ is intrinsic to f .

Definition 5.68. A function $F : X^{\text{an}} \rightarrow \mathbf{R}$ is *piecewise linear* provided that for any geodesic segment $\alpha : [a, b] \hookrightarrow \mathbf{H}(X^{\text{an}})$ the pullback $F \circ \alpha : [a, b] \rightarrow \mathbf{R}$ is piecewise linear. The *outgoing slope* of a piecewise linear function F at a point $x \in \mathbf{H}(X^{\text{an}})$ along a tangent direction $v \in T_x$ is defined to be

$$d_v F(x) = \lim_{\varepsilon \rightarrow 0} (F \circ \alpha)'(\varepsilon)$$

where $\alpha : [0, a] \hookrightarrow X^{\text{an}}$ is a nontrivial geodesic segment starting at x which represents v . We say that a piecewise linear function F is *harmonic* at a point $x \in X^{\text{an}}$ provided that the outgoing slope $d_v F(x)$ is nonzero for only finitely many $v \in T_x$, and $\sum_{v \in T_x} d_v F(x) = 0$. We say that F is *harmonic* if it is harmonic for all $x \in \mathbf{H}(X^{\text{an}})$.

Theorem 5.69. (Slope Formula) *Let f be an algebraic function on X with no zeros or poles and let $F = -\log |f| : X^{\text{an}} \rightarrow \mathbf{R}$. Let V be a semistable vertex set of X and let $\Sigma = \Sigma(X, V)$. Then:*

- (1) $F = F \circ \tau_\Sigma$ where $\tau_\Sigma : X^{\text{an}} \rightarrow \Sigma$ is the retraction.
- (2) F is piecewise linear with integer slopes, and F is linear on each edge of Σ .
- (3) If x is a type-2 point of X^{an} and $v \in T_x$ then $d_v F(x) = \text{ord}_v(\tilde{f}_x)$.
- (4) F is harmonic.
- (5) Let $x \in D$, let e be the ray in Σ whose closure in \widehat{X} contains x , let $y \in V$ be the other endpoint of e , and let $v \in T_y$ be the tangent direction represented by e . Then $d_v F(y) = \text{ord}_x(f)$.

Proof. The first claim follows from Lemma 5.13 and the fact that a unit on an open ball has constant absolute value. The linearity of F on edges of Σ is Proposition 5.10. Since $F = F \circ \tau_\Sigma$ we have that F is constant in a neighborhood of any point of type 4, and any geodesic segment contained in $\mathbf{H}_o(X^{\text{an}})$ is contained in a skeleton by Corollary 5.64, so F is piecewise linear. The last claim is Proposition 5.10(2). The harmonicity of F is proved as follows: if $x \in X^{\text{an}}$ has type 4 then x has one tangent direction and F is locally constant in a neighborhood of x , so $\sum_{v \in T_x} d_v F(x) = 0$. If x has type 3 then x is contained in the interior of an edge e of a skeleton, and the two tangent directions v, w at x are represented by the two paths emanating from x in e ; since F is linear on e we have $d_v F(x) = -d_w F(x)$. The harmonicity of F at type-2 points is an immediate consequence of (3) and the fact that the divisor of a meromorphic function on a smooth complete curve has degree zero.

The heart of this theorem is (3), which again is essentially a result of Bosch and Lütkebohmert. Let x be a type-2 point of X^{an} , let U be a simple neighborhood of x , and let \mathfrak{X} be a semistable formal model of \widehat{X} such that $x \in V(\mathfrak{X})$ and $U = \text{red}^{-1}(\overline{\mathcal{C}})$ where $\overline{\mathcal{C}}$ is the smooth irreducible component of $\overline{\mathfrak{X}}$ with generic point $\text{red}(x)$. We may and do assume that $V(\mathfrak{X})$ is a semistable vertex set of X containing V . Let $\overline{\mathcal{C}}' \subset \overline{\mathcal{C}}$ be the affine curve obtained by deleting all points $\xi \in \overline{\mathcal{C}}$ which are not smooth in $\overline{\mathfrak{X}}$ and let \mathcal{C}' be the induced formal affine subscheme of \mathfrak{X} . Then $(\mathcal{C}')^{\text{an}} = \text{red}^{-1}(\overline{\mathcal{C}}') = \tau_{V(\mathfrak{X})}^{-1}(x)$ is an affinoid domain in X^{an} with Shilov boundary $\{x\}$. If we scale f such that $|f(x)| = 1$ then f and f^{-1} both have

supremum norm 1 on $\tau_{V(\mathfrak{X})}^{-1}(x)$. It follows that the residue \tilde{f}_x of f is a unit on $\overline{\mathfrak{C}}'$, so $\text{ord}_\zeta(\tilde{f}_x) = 0$ for all $\zeta \in \overline{\mathfrak{C}}'(k)$. By (1) we have that F is constant on $\tau_{V(\mathfrak{X})}^{-1}(x)$, so $d_v F(x) = \text{ord}_v(\tilde{f}_x) = 0$ for all $v \in T_x$ corresponding to closed points of $\overline{\mathfrak{C}}'$.

Now let $v \in T_x$ correspond to a point $\xi \in \overline{\mathfrak{C}}$ which is contained in two irreducible components $\overline{\mathfrak{C}}, \overline{\mathfrak{D}}$ of $\overline{\mathfrak{X}}$. Let $y \in X^{\text{an}}$ be the point reducing to the generic point of $\overline{\mathfrak{D}}$ and let e be the edge in $\Sigma(X, V(\mathfrak{X}))$ connecting x and y , so e is a geodesic segment representing v . If e° is the interior of e then $A = \tau_{V(\mathfrak{X})}^{-1}(e^\circ) = \text{red}^{-1}(\xi)$ in an open annulus; we let r be the modulus of A . By [BL85, Proposition 3.2] we have $F(x) - F(y) = -r \cdot \text{ord}_\xi(\tilde{f}_x)$. Since F is linear on e we also have $F(x) - F(y) = -r \cdot d_v F(x)$, whence the desired equality. \blacksquare

Remark 5.70. Theorem 5.69 will be one of our main tools in §§7–8. It is also proved in [Thu05, Proposition 3.3.15], in the following form: if f is a nonzero meromorphic function on \widehat{X}^{an} , then the extended real-valued function $\log |f|$ on X satisfies the differential equation

$$(5.70.1) \quad dd^c \log |f| = \delta_{\text{div}(f)}$$

where dd^c is a distribution-valued operator which serves as a nonarchimedean analogue of the classical dd^c -operator on a Riemann surface. One can regard (5.70.1) as a nonarchimedean analogue of the classical ‘Poincaré-Lelong formula’ for Riemann surfaces. Since it would lead us too far astray to recall the general definition of Thuillier’s dd^c -operator on an analytic curve, we simply call Theorem 5.69 the Slope Formula.

Remark 5.71.

- (1) See [BR10, Example 5.20] for a version of Theorem 5.69 for $X = \mathbf{P}^1$.
- (2) It is an elementary exercise that conditions (4) and (5) of Theorem 5.69 uniquely determine the function $F : \Sigma \rightarrow \mathbf{R}$ up to addition by a constant; see the proof of [BR10, Proposition 3.2(A)].

6. THE TROPICALIZATION OF A NONARCHIMEDEAN ANALYTIC CURVE

6.1. The setup. Throughout this section X denotes a smooth connected algebraic curve realized as a closed subscheme of a torus \mathbf{T} , \widehat{X} is the smooth completion of X , and $D = \widehat{X}(K) \setminus X(K)$ is the set of punctures. We will denote a choice of semistable vertex set for X by V , and we let $\Sigma = \Sigma(X, V)$ be the associated skeleton.

If we choose a basis for M , then we obtain isomorphisms $N_{\mathbf{R}} \cong \mathbf{R}^n$ and $K[\mathbf{T}] \cong K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$; if $f_i \in K[X]^\times$ is the image of x_i then

$$(6.1.1) \quad \text{trop}(\|\cdot\|) = (-\log \|f_1\|, \dots, -\log \|f_n\|).$$

6.2. Compatible polyhedral structures. The tropicalization of X is a polyhedral complex of pure dimension 1 in $N_{\mathbf{R}}$. We can regard $\text{Trop}(X)$ as a dimension-1 abstract G -rational polyhedral complex where the metric on the edges is given by the lattice length, i.e. the length in the direction of a primitive lattice vector in N .

Recall the following consequence of the Slope Formula (5.69):

Lemma 6.3. *Let e be an edge of Σ and let $f \in K[X]^\times$. The map $\|\cdot\| \mapsto -\log \|f\| : X^{\text{an}} \rightarrow \mathbf{R}$ restricts to a G -affine linear function from e to \mathbf{R} with integer slope.*

Since the G -rational points of an edge e of Σ are exactly the type-2 points of X^{an} contained in e (cf. (5.26)), it follows from (6.1.1) and the G -rationality of $-\log \|f\|$ as above that trop maps type-2 points into N_G .

Proposition 6.4.

- (1) *The map $\text{trop} : X^{\text{an}} \rightarrow \text{Trop}(X)$ factors through the retraction $\tau_\Sigma : X^{\text{an}} \rightarrow \Sigma$.*

- (2) We can choose V and a polyhedral complex structure on $\text{Trop}(X)$ as in Theorem 4.29 such that $\text{trop} : \Sigma \rightarrow \text{Trop}(X)$ is a morphism of dimension-1 abstract G -rational polyhedral complexes.

Proof. The first part follows from (6.1.1) and the Slope Formula (5.69). Let e be an edge of Σ . It follows from Lemma 6.3 as applied to f_1, \dots, f_n that trop restricts to an expansion by an integer multiple with respect to the intrinsic metric on e and the lattice length on its image. Hence there exist refinements of the polyhedral structures on Σ and on $\text{Trop}(X)$ such that $\text{trop} : \Sigma \rightarrow \text{Trop}(X)$ becomes a morphism of dimension-1 abstract G -rational polyhedral complexes. By Proposition 5.27(2) any refinement of Σ is also a skeleton of X . \blacksquare

From now on we assume that our skeleton Σ of X and our choice of polyhedral structure on $\text{Trop}(X)$ are compatible in the sense of Proposition 6.4(2). If $e \subset \Sigma$ is an interval contained in an edge we let $\ell_{\text{an}}(e)$ be its length with respect to the skeletal metric, and if $e' \subset \text{Trop}(X)$ is an interval contained in an edge we let $\ell_{\text{Trop}}(e')$ be its lattice length. As a consequence of Proposition 6.4(2), if $\text{trop}(e) = e'$ then $\ell_{\text{Trop}}(e')$ is an integer multiple of $\ell_{\text{an}}(e)$.

Definition 6.5. Let $e \subset \Sigma$ be an edge and let $e' \subset \text{Trop}(X)$ be its image. We define the *expansion factor* of e to be the unique integer $m_{\text{rel}}(e) \in \mathbf{Z}_{\geq 0}$ such that

$$\ell_{\text{Trop}}(\text{trop}(\tilde{e})) = m_{\text{rel}}(e) \cdot \ell_{\text{an}}(\tilde{e})$$

for any finite-length segment \tilde{e} contained in e .

Remark 6.6. For $u \in M$ let $f_u \in K[X]^\times$ be the image of the character $x^u \in K[M]$. Let e be an edge of Σ and let $s_u \in \mathbf{Z}_{\geq 0}$ be the absolute value of the slope of $-\log \|f_u\|$ on e . It follows easily from the definitions that $m_{\text{rel}}(e) = \gcd\{s_u : u \in M\}$. More concretely, let u_1, \dots, u_n be a basis for M and let $f_i \in K[X]^\times$ be the image of x^{u_i} , so $\text{trop}(\|\cdot\|) = (-\log \|f_1\|, \dots, -\log \|f_n\|)$. Let $s_i \in \mathbf{Z}_{\geq 0}$ be the absolute value of the slope of $-\log \|f_i\|$ on e . Then

$$(6.6.1) \quad m_{\text{rel}}(e) = \gcd(s_1, \dots, s_n).$$

6.7. We now come to one of the key results of this section. Let $e \subset \Sigma$ be a bounded edge,⁸ and assume that $e' = \text{trop}(e)$ is an edge of $\text{Trop}(X)$ (as opposed to a vertex). The inverse image of the interior of e' under trop is a disjoint union of open annuli, one of whose skeleta is the interior of e . Hence if $x \in e$ is the unique point mapping to some $w \in \text{relint}(e') \cap N_G$ then $\mathcal{Y}_x := \tau_\Sigma^{-1}(x) \cong \mathbf{S}(1)$ is a connected component of $\mathcal{X}^w = \mathcal{U}^w \cap X^{\text{an}}$ (Definition 4.14). Let W be the affine span of e' , let \mathbf{T}' be the torus transverse to W (4.27), let \mathbf{T}'' be the torus parallel to W (Remark 4.28), and choose a splitting $\mathbf{T} \rightarrow \mathbf{T}''$. We have a finite surjective morphism $\mathcal{X}^w \rightarrow \mathcal{U}^{w''}$ by Theorem 4.31, where w'' is the image of w in N_G'' .

For $y \in e$ the relative multiplicity $m_{\text{rel}}(y)$ was defined in (4.21).

Theorem 6.8. (Compatibility of multiplicities) *With the above notation,*

$$m_{\text{rel}}(e) = [\mathcal{Y}_x : \mathcal{U}^{w''}] = m_{\text{rel}}(x).$$

Moreover $m_{\text{rel}}(e) = m_{\text{rel}}(y)$ for any y in the interior of e (even if $\text{trop}(y) \notin N_G$).

Proof. Let P be a G -rational closed interval contained in the interior of e' and containing w . Then

$$\mathcal{Y} := \tau_\Sigma^{-1}(\text{trop}^{-1}(P) \cap e) \cong \mathbf{S}(a, b)$$

is a connected component of \mathcal{X}^P ; it is a closed annulus of nonzero modulus with skeleton $\text{trop}^{-1}(P) \cap e$. Let P'' be the image of P in N_G'' , so $\ell_{\text{Trop}}(P'') = \ell_{\text{Trop}}(P)$. By Theorem 4.31 the morphism $\mathcal{Y} \rightarrow \mathcal{U}^{P''}$ is finite and surjective, and $[\mathcal{Y} : \mathcal{U}^{P''}] = [\mathcal{Y}_x : \mathcal{U}^{w''}]$ by Proposition 3.30(1). The torus

⁸If e is an infinite ray, we can compute $m_{\text{rel}}(e)$ by refining the polyhedral structure on Σ so that $m_{\text{rel}}(e)$ coincides with $m_{\text{rel}}(\tilde{e})$ for some bounded edge \tilde{e} in the refinement. Thus we can assume without loss of generality in Theorem 6.8 that e is bounded.

\mathbf{T}'' is one-dimensional, so $\mathcal{U}^{P''} = \text{trop}^{-1}(P'')$ is an annulus and P'' is by definition the skeleton of $\mathcal{U}^{P''}$. We have

$$m_{\text{rel}}(e) \cdot \ell_{\text{an}}(\text{trop}^{-1}(P) \cap e) = \ell_{\text{Trop}}(P) = \ell_{\text{Trop}}(P'') = [\mathcal{Y} : \mathcal{U}^{P''}] \cdot \ell_{\text{an}}(\text{trop}^{-1}(P) \cap e),$$

where the final equality is by Corollary 5.6. Since x is the unique Shilov boundary point of $\mathcal{Y}_x \cong \mathbf{S}(1)$, the equality $[\mathcal{Y}_x : \mathcal{U}^{w''}] = m_{\text{rel}}(x)$ is a consequence of Proposition 4.32.

By a standard argument involving extension of the ground field (cf. Lemma 4.22 and (4.18)), the second statement follows from the first. \blacksquare

Corollary 6.9. *Fix an edge e' of $\text{Trop}(X)$ and let e_1, \dots, e_r be the edges of Σ mapping homeomorphically onto e' . Then*

$$m_{\text{Trop}}(e') = \sum_{i=1}^r m_{\text{rel}}(e_i).$$

Proof. This follows immediately from Theorem 6.8 and Proposition 4.24. \blacksquare

Remark 6.10. With the notation in Corollary 6.9, let w be a G -rational point contained in the relative interior of e' . The affinoid space $\text{trop}^{-1}(w)$ is isomorphic to $\coprod_{i=1}^r \mathbf{S}(1)$ as in (6.7). Since the canonical reduction of $\mathbf{S}(1)$ is isomorphic to \mathbf{G}_m , the integer r is equal to the number of irreducible components in the canonical reduction of \mathcal{X}^w .

Corollary 6.11. *If e' is an edge of $\text{Trop}(X)$ and $m_{\text{Trop}}(e') = 1$, then there is a unique edge e in Σ such that trop maps e homeomorphically and isometrically onto e' . The edge e is in fact the unique geodesic segment (or ray) in X^{an} which is mapped homeomorphically by trop onto e' .*

Corollary 6.12. *Let $x \in \mathbf{H}(X^{\text{an}})$. Then $m_{\text{rel}}(x) > 0$ if and only if x belongs to an edge of Σ mapping homeomorphically onto its image via trop .*

Proof. Suppose that x is contained in an edge $e \subset \Sigma$ mapping homeomorphically onto its image $e' = \text{trop}(e)$. If x is in the interior of e then $m_{\text{rel}}(x) = m_{\text{rel}}(e) > 0$ by Theorem 6.8. Otherwise $w = \text{trop}(x) \in N_G$, and x is contained in the limit boundary $\partial_{\text{lim}} \mathcal{X}^w$ of $\mathcal{X}^w = \text{trop}^{-1}(w)$ because it is a limit of points of e which are not contained in \mathcal{X}^w . Since $\partial_{\text{lim}} \mathcal{X}^w$ is the Shilov boundary of \mathcal{X}^w , by definition we have $m_{\text{rel}}(x) > 0$.

Now suppose that x is not contained in an edge of Σ mapping homeomorphically onto its image. If x has type 4 then $m_{\text{rel}}(x) = 0$ by definition. Otherwise by Corollary 5.56 we can enlarge Σ if necessary to assume that $x \in \Sigma$. Recall that every edge of Σ maps homeomorphically onto its image or is crushed to a vertex of $\text{Trop}(X)$. By hypothesis all edges containing x are crushed to a vertex of $\text{Trop}(X)$, so $w = \text{trop}(x) \in N_G$. Hence there is an open neighborhood U of x in Σ contained in \mathcal{X}^w . Then $\tau_{\Sigma}^{-1}(U)$ is a neighborhood of x in X^{an} contained in \mathcal{X}^w , so $x \notin \partial \mathcal{X}^w$ and hence $m_{\text{rel}}(x) = 0$. \blacksquare

6.13. Slopes as orders of vanishing. The Slope Formula (5.69) provides a useful interpretation of the quantities s_i appearing in Remark 6.6 in terms of orders of vanishing. Assume that V is a strongly semistable vertex set of \widehat{X} (in addition to being a semistable vertex set of X). Let \mathfrak{X} be the strongly semistable formal model of \widehat{X} associated to V (Theorem 5.38), let $x \in V$, and let $\overline{\mathfrak{c}}$ be the irreducible component of $\overline{\mathfrak{X}}$ whose generic point is $\text{red}(x)$. Let e be an edge of Σ adjacent to x and let $\xi \in \overline{\mathfrak{c}}(k)$ be the reduction of the interior of e . Theorem 5.69 says that if f is a nonzero rational function on \widehat{X} then the slope s of $-\log|f|$ along e (in the direction away from x) is equal to $\text{ord}_{\xi}(\tilde{f}_x)$. One can use this fact to give a simple proof of the well-known balancing formula for tropical curves:

Theorem 6.14. (The balancing formula for tropical curves) *Let w be a vertex of $\text{Trop}(X)$ and let $\vec{v}_1, \dots, \vec{v}_t$ be the primitive integer tangent directions at w corresponding to the various edges e'_1, \dots, e'_t incident to w . Then $\sum_{j=1}^t m_{\text{Trop}}(e'_j) \vec{v}_j = 0$.*

Proof. We use the setup in (6.13). Let $f_1, \dots, f_n \in K[X]^{\times}$ be the coordinate functions as in (6.6) and let $F_i = -\log|f_i|$. Let $x \in \Sigma \cap \text{trop}^{-1}(w)$ be a vertex. By Theorem 5.69 we have $0 = \sum_{v \in T_x} d_v F_i(x)$ for each $i = 1, \dots, n$. Since we are assuming that V is a strongly semistable vertex set,

each tangent direction $v \in T_x$ along which some F_i has nonzero slope is represented by a unique edge $e_v = [x, y_v]$ of Σ adjoining x , and $d_v F_i(x)$ is just the slope of $-\log |f_i|$ along e_v . If $\text{trop}(e_v) = \{w\}$ then $d_v F_i(x) = 0$ for all i , and otherwise

$$d_v F_i(x) = \frac{\log |f_i(x)| - \log |f_i(y_v)|}{\ell_{\text{an}}(e_v)} = m_{\text{rel}}(e_v) \frac{\log |f_i(x)| - \log |f_i(y_v)|}{\ell_{\text{Trop}}(\text{trop}(e_v))}.$$

By Corollary 6.9, for each i we have

$$\begin{aligned} \sum_{j=1}^t m_{\text{Trop}}(e'_j)(\vec{v}_j)_i &= \sum_{j=1}^t \left(\sum_{e=[x,y] \xrightarrow{\sim} e'_j} m_{\text{rel}}(e) \right) \frac{\log |f_i(x)| - \log |f_i(y)|}{\ell_{\text{Trop}}(e'_j)} \\ &= \sum_{x \mapsto w} \sum_{v \in T_x} d_v F_i(x) = 0, \end{aligned}$$

which implies the result. ■

6.15. Faithful representations. If Y_Δ is a proper toric variety with dense torus \mathbf{T} then $X \hookrightarrow \mathbf{T}$ extends in a unique way to a morphism $\iota : \widehat{X} \rightarrow Y_\Delta$, which is a closed immersion for suitable Y_Δ . The intersection $\widehat{X}^{\text{an}} \cap (Y_\Delta^{\text{an}} \setminus \mathbf{T}^{\text{an}})$ is the finite set of type-1 points $D = \widehat{X}^{\text{an}} \setminus X^{\text{an}}$. We write $\text{trop}(\iota) : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta)$ for the induced tropicalization map, and we set $\text{Trop}(\widehat{X}, \iota) = \text{trop}(\iota)(\widehat{X}^{\text{an}}) \subset N_{\mathbf{R}}(\Delta)$.

6.15.1. Let $Y_\Delta, Y_{\Delta'}$ be toric varieties with dense tori \mathbf{T}, \mathbf{T}' and let $\iota : \widehat{X} \hookrightarrow Y_\Delta$ and $\iota' : \widehat{X} \hookrightarrow Y_{\Delta'}$ be closed immersions whose images meet the dense torus. We say that ι' *dominates* ι and we write $\iota' \geq \iota$ provided that there exists a morphism $\psi : Y_{\Delta'} \rightarrow Y_\Delta$ of toric varieties such that $\psi \circ \iota' = \iota$. In this case we have an induced morphism $\text{Trop}(\psi) : \text{Trop}(\widehat{X}, \iota') \rightarrow \text{Trop}(\widehat{X}, \iota)$ making the triangle

$$\begin{array}{ccc} & \widehat{X}^{\text{an}} & \\ \text{trop}(\iota') \swarrow & & \searrow \text{trop}(\iota) \\ \text{Trop}(\widehat{X}, \iota') & \xrightarrow{\text{Trop}(\psi)} & \text{Trop}(\widehat{X}, \iota) \end{array}$$

commute. Since $\text{trop}(\iota)$ and $\text{trop}(\iota')$ are surjective, the map $\text{Trop}(\psi)$ is independent of the choice of ψ , so the set of ‘tropicalizations of toric embeddings’ is a filtered inverse system.

6.15.2. By a *finite subgraph* of \widehat{X}^{an} we mean a *connected* compact subgraph of a skeleton of \widehat{X} . Any finite union of geodesic segments in $\mathbf{H}_o(\widehat{X}^{\text{an}})$ is contained in a skeleton by Corollary 5.64, so we can equivalently define a finite subgraph of \widehat{X}^{an} to be an isometric embedding of a finite connected metric graph Γ into $\mathbf{H}_o(\widehat{X}^{\text{an}})$. Let $\widehat{X} \hookrightarrow Y_\Delta$ be a closed immersion into a toric variety with dense torus \mathbf{T} such that $\widehat{X} \cap \mathbf{T} \neq \emptyset$. We say that a finite subgraph Γ of \widehat{X}^{an} is *faithfully represented* by $\text{trop} : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta)$ if trop maps Γ homeomorphically and isometrically onto its image Γ' (which is contained in $N_{\mathbf{R}}(\Delta)$). We say that trop is *faithful* if it faithfully represents a skeleton Σ of \widehat{X} .

Remark 6.16. When considering a closed connected subset Γ of $\mathbf{H}(X^{\text{an}})$ or Γ' of $\text{Trop}(X)$, we will always implicitly endow it with the shortest-path metric. In general this is *not* the same as the metric on Γ (resp. Γ') induced by the (shortest-path) metric on $\mathbf{H}(X^{\text{an}})$ (resp. $\text{Trop}(X)$). With this convention, Γ (resp. Γ') is a *length space* in the sense of [Pap05, Definition 2.1.2], so any homeomorphism $\Gamma \rightarrow \Gamma'$ which is a local isometry is automatically an isometry by Corollary 3.4.6 of loc. cit. This will be used several times in what follows.

The following result shows that if Γ is faithfully represented by a given tropicalization, then it is also faithfully represented by all ‘larger’ tropicalizations.

Lemma 6.17. *Let $\iota : \widehat{X} \hookrightarrow Y_\Delta$ and $\iota' : \widehat{X} \hookrightarrow Y_{\Delta'}$ be closed immersions of \widehat{X} into toric varieties whose images meet the dense torus and such that $\iota' \geq \iota$. If a finite subgraph Γ of \widehat{X}^{an} is faithfully represented by $\text{trop}(\iota) : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta)$ then Γ is faithfully represented by $\text{trop}(\iota') : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta')$.*

Proof. Without loss of generality, we may replace \widehat{X} by $X = \widehat{X} \setminus D$ and assume that $Y_{\Delta'}$ and Y_{Δ} are tori with $\iota = (f_1, \dots, f_n)$ and $\iota' = (f_1, \dots, f_m)$ for some $m \geq n$. The result is now clear from (6.6.1). ■

We will show in Theorem 6.20 below that any finite subgraph of \widehat{X}^{an} is faithfully represented by some tropicalization. First we need two lemmas.

Lemma 6.18. *Let e be an edge of a skeleton Σ of \widehat{X} with distinct endpoints x, y . There exists a nonzero meromorphic function f on \widehat{X} such that $F = -\log |f|$ has the following properties:*

- (1) $F \geq 0$ on e , and $F(x) = F(y) = 0$.
- (2) *There exist (not necessarily distinct) type-2 points x', y' in the interior e° of e such that $\rho(x, x') = \rho(y, y')$, such that F has slope ± 1 on $[x, x']$ and $[y, y']$, and such that F is constant on $[x', y']$, as shown in Figure 6.*

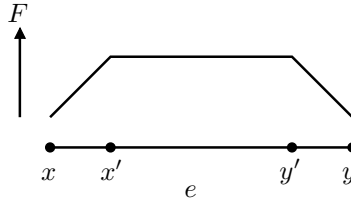


FIGURE 6. The graph of the function $F = -\log |f| : e \rightarrow \mathbf{R}_{\geq 0}$ constructed in Lemma 6.18.

Proof. We may assume without loss of generality that $\Sigma = \Sigma(\widehat{X}, V(\mathfrak{X}))$ for a strongly semistable formal model \mathfrak{X} of \widehat{X} . For each irreducible component $\overline{\mathfrak{C}}_\nu$ of $\overline{\mathfrak{X}}$, a simple argument using the Riemann-Roch theorem allows us to choose a rational function \tilde{f}_ν on $\overline{\mathfrak{C}}_\nu$ which vanishes to order 1 at every singular point of $\overline{\mathfrak{X}}$ lying on $\overline{\mathfrak{C}}_\nu$. By [BL85, Corollary 3.8] there exists a nonzero rational function f on \widehat{X} whose poles all reduce to smooth points of $\overline{\mathfrak{X}}$ and which induces the rational function \tilde{f}_ν on each irreducible component $\overline{\mathfrak{C}}_\nu$ of $\overline{\mathfrak{X}}$ (the gluing condition from loc. cit. is trivially in this situation). The function f constructed in the proof of loc. cit. is defined on an affinoid domain U of \widehat{X}^{an} containing x and y , and \tilde{f}_x, \tilde{f}_y are the restrictions of the residue of f in the canonical reduction of U . Therefore we have $|f(x)| = |f(y)| = 1$. Since $\{x, y\}$ is the Shilov boundary of $\tau_\Sigma^{-1}(e)$, this proves (1).

By Theorem 5.69(3) the outgoing slope of F at x or y in the direction of e is 1. Let ξ be the singular point of $\overline{\mathfrak{X}}$ whose formal fiber is $\tau_\Sigma^{-1}(e^\circ)$. Since f has no poles on the formal fibers above singular points, f restricts to an analytic function on the open annulus $\tau_\Sigma^{-1}(e^\circ)$. Part (2) now follows from (1) and Proposition 5.10(1). ■

Lemma 6.19. *Let $A \subset \widehat{X}^{\text{an}}$ be an affinoid domain isomorphic to a closed annulus $\mathbf{S}(a)$ with nonzero modulus. There exists a nonzero meromorphic function f on \widehat{X} such that $F = -\log |f|$ is linear with slope ± 1 on $\Sigma(A)$.*

Proof. Choose an identification of A with $\mathbf{S}(a) = \mathcal{M}(K\langle at^{-1}, t \rangle)$. By [FM86, Théorème 4, §2.4] the ring of meromorphic functions on \widehat{X} which are regular on A is dense in $K\langle at^{-1}, t \rangle$. Hence there exists a meromorphic function f on \widehat{X} such that $f \in K\langle at^{-1}, t \rangle$ and $|f - t|_{\text{sup}} < 1$. It follows from Proposition 5.2 that f is also a parameter for the annulus A , so $-\log |f|$ is linear with slope ± 1 on $\Sigma(A)$. ■

Theorem 6.20. *If Γ is any finite subgraph of \widehat{X}^{an} then there is a closed immersion $\widehat{X} \hookrightarrow Y_\Delta$ of \widehat{X} into a quasiprojective toric variety Y_Δ such that $\text{trop} : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta)$ faithfully represents Γ . In particular, there exists a faithful tropicalization.*

Proof. Since Γ is by definition contained in a skeleton Σ , we may assume that $\Gamma = \Sigma$. Taking a refinement of Σ if necessary, we assume without loss of generality that Σ does not have any loop

edges. We claim that after possibly refining Σ further, for each edge $e \subset \Sigma$ there exists a nonzero meromorphic function f on \widehat{X} such that $\log |f|$ has slope ± 1 on e .

Let $e = [x, y]$ be an edge of Σ , and let f and $x', y' \in e^\circ$ be as in Lemma 6.18. Then $[x, x']$ and $[y', y]$ are edges in a refinement of Σ , and $\log |f|$ has slope ± 1 on $[x, x']$ and $[y', y]$. If $x' = y'$ then we are done with e ; otherwise we let $e' = [x', y']$. By construction $e' \subset e^\circ$, so $\tau_\Sigma^{-1}(e')$ is a closed annulus of nonzero modulus, and we may apply Lemma 6.19 to find f' such that $\log |f'|$ has slope ± 1 on e' . This proves the claim.

By (6.6.1), if $\Phi = \{f_1, \dots, f_r\}$ is any collection of meromorphic functions on X such that (a) for each edge e of Σ there is an i such that $\log |f_i|$ has slope ± 1 on e , and (b) $\varphi = (f_1, \dots, f_r)$ induces a closed immersion of a dense open subscheme X of \widehat{X} into a torus $\mathbf{T} \cong \mathbf{G}_m^r$, then $\text{trop} \circ \varphi$ maps each edge of Σ isometrically onto its image. Since φ extends to a closed immersion $\widehat{X} \hookrightarrow Y_\Delta$ into a suitable compactification Y_Δ of \mathbf{T} , it only remains to show that we can enlarge Φ so that $\varphi|_\Sigma$ is injective; then $\text{trop} \circ \varphi$ maps Σ isometrically onto its image by Remark 6.16.

Let e be an edge of Σ . Since Σ has at least two edges, (5.42.1) shows that $\tau_\Sigma^{-1}(e)$ is an affinoid domain in \widehat{X}^{an} . By [FM86, Théorème 1, §1.4], there is a meromorphic function f on \widehat{X} such that $\tau_\Sigma^{-1}(e) = \{x \in \widehat{X}^{\text{an}} : |f(x)| \leq 1\}$. Adding such an f to Φ for every edge e , we may assume that $\text{trop} \circ \varphi$ is injective on $\Sigma \setminus V$, i.e., that if $\text{trop}(\varphi(x)) = \text{trop}(\varphi(y))$ for $x, y \in \Sigma$ then x, y are vertices. By the definition of \widehat{X}^{an} , if $x, y \in \widehat{X}^{\text{an}}$ are distinct points then there exists a meromorphic function f on \widehat{X}^{an} such that $|f(x)| \neq |f(y)|$. Adding such f to Φ for every pair of vertices yields a faithful tropicalization. ■

We obtain the following theorem as a consequence:

Theorem 6.21. *Let Γ be a finite subgraph of \widehat{X} . Then there exists a quasiprojective toric embedding $\iota : \widehat{X} \hookrightarrow Y_\Delta$ such that for every quasiprojective toric embedding $\iota' : \widehat{X} \hookrightarrow Y_{\Delta'}$ with $\iota' \geq \iota$, the tropicalization map $\text{trop}(\iota') : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta')$ maps Γ homeomorphically and isometrically onto its image.*

Proof. By Theorem 6.20, there exists a closed embedding ι such that $\text{trop}(\iota)$ maps Γ homeomorphically and isometrically onto its image. By Lemma 6.17, the same property holds for any closed embedding $\iota' \geq \iota$. ■

As mentioned in the introduction, Theorem 6.21 can be interpreted colloquially as saying that the homeomorphism in Theorem 1.2 is an isometry.

With a little more work, we obtain the following strengthening of Theorem 6.21 in which the finite metric graph Γ is replaced by an arbitrary skeleton of X (which is no longer required to be compact or of finite length).

Theorem 6.22. *Let Σ be any skeleton of X . Then there exists a quasiprojective toric embedding $\iota : X \hookrightarrow Y_\Delta$ such that for every quasiprojective toric embedding $\iota' : X \hookrightarrow Y_{\Delta'}$ with $\iota' \geq \iota$, the tropicalization map $\text{trop}(\iota') : \widehat{X}^{\text{an}} \rightarrow N_{\mathbf{R}}(\Delta')$ maps Σ homeomorphically and isometrically onto its image.*

Proof. Using Lemma 6.17, it suffices to prove that there exists a closed embedding ι such that $\text{trop}(\iota)$ maps Σ homeomorphically and isometrically onto its image.

For each point $p \in D = \widehat{X} \setminus X$, choose a pair of relatively prime integers $m_1(p), m_2(p)$ bigger than $2g$, where g is the genus of \widehat{X} . By the Riemann-Roch theorem, there are rational functions $f_1^{(p)}$ and $f_2^{(p)}$ on \widehat{X} such that $f_i^{(p)}$ has a pole of exact order $m_i(p)$ at p and no other poles for $i = 1, 2$. Let U_p be an (analytic) open neighborhood of p on which $f_1^{(p)}$ and $f_2^{(p)}$ have no zeros and let U be the union of U_p for all $p \in D$.

Let $\Gamma = \Sigma \setminus (\Sigma \cap U)$. Then Γ is a finite subgraph of X so by Theorem 6.20, there exists a closed embedding ι_0 such that $\text{trop}(\iota_0)$ maps Γ homeomorphically and isometrically onto its image. We can choose the U_p such that the complement $\Sigma \setminus \Gamma$ consists of finitely many open infinite rays r_p , one for each point $p \in D$. By the Slope Formula, the absolute value of the slope of $\log |f_i^{(p)}|$ along r_p is $m_i(p)$. Since $\gcd(m_1(p), m_2(p)) = 1$ for all $p \in D$, if we enhance the embedding ι_0 to a larger embedding ι by adding the coordinate functions $f_1^{(p)}$ and $f_2^{(p)}$ for all $p \in D$, $\text{trop}(\iota)$ has multiplicity one along each

ray r_p by Remark 6.6. By Lemma 6.17, $\text{trop}(\iota)$ also has multiplicity one at every edge of Γ . It follows easily (as in the proof of Theorem 6.20) that $\text{trop}(\iota)$ maps Σ homeomorphically and isometrically onto its image as desired. \blacksquare

6.23. Certifying faithfulness. It is useful to be able to certify that a given tropicalization map is faithful using only ‘tropical’ computations.

Theorem 6.24. *Let Γ' be a compact connected subset of $\text{Trop}(X)$ and suppose that $m_{\text{Trop}}(w) = 1$ for all $w \in \Gamma' \cap N_G$. Then there is a unique closed subset $\Gamma \subset \mathbf{H}_o(X^{\text{an}})$ mapping homeomorphically onto Γ' , and this homeomorphism is an isometry.*

Proof. Since m_{Trop} is constant along the interior of each edge of $\text{Trop}(X)$ by Theorem 4.29(1) (for points not contained in N_G this is proved by a standard ground field extension argument) and Γ' is a finite union of closed intervals, we have $m_{\text{Trop}}(w) = 1$ for all $w \in \Gamma'$. By Proposition 4.24, for each $w \in \Gamma'$ there is a unique point $x = x_w \in \mathbf{H}_o(X^{\text{an}})$ such that $\text{trop}(x) = w$ and $m_{\text{rel}}(x) > 0$. Let $\Gamma = \{x_w : w \in \Gamma'\}$. The natural continuous map $\text{trop} : \Gamma \rightarrow \Gamma'$ is bijective. It follows from Corollaries 6.11 and 6.12 that Γ is also a finite union of closed intervals, hence compact. Thus $\text{trop} : \Gamma \rightarrow \Gamma'$, being a continuous bijection between compact Hausdorff spaces, is a homeomorphism. By Corollary 6.11 this homeomorphism is an isometry.

As for the uniqueness of Γ , let $\tilde{\Gamma}$ be any closed subset of $\mathbf{H}_o(X^{\text{an}})$ mapping homeomorphically onto Γ' . Fix $w \in \Gamma'$, and let x be the point in $\tilde{\Gamma}$ with $\text{trop}(x) = w$. Since x belongs to a closed segment of X^{an} mapping homeomorphically onto its image via trop (namely the inverse image in $\tilde{\Gamma}$ of an edge in $\text{Trop}(X)$ containing w), it follows from Corollary 6.12 that $m_{\text{rel}}(x) > 0$. Hence $x = x_w$, so $\tilde{\Gamma} = \Gamma$. \blacksquare

In order to apply Theorem 6.24, it is useful to know that one can sometimes determine the multiplicity at a point $w \in \text{Trop}(X) \cap N_G$ just from the local structure of $\text{Trop}(X)$ at w , i.e., from the combinatorics of $\text{Star}(w)$. Recall that if $\vec{v}_1, \dots, \vec{v}_r$ are the primitive generators of the edge directions in $\text{Trop}(X)$ at w , and a_i is the tropical multiplicity of the edge corresponding to \vec{v}_i , then the balancing condition says that $a_1\vec{v}_1 + \dots + a_r\vec{v}_r = 0$. Now, if Z_1, \dots, Z_s are the irreducible components of $\text{in}_w(X)$ then the tropicalization of each Z_j (as a subscheme of the torus torsor $\overline{\mathbf{T}}^w$ over the trivially-valued field k) is a union of rays spanned by a subset of $\{\vec{v}_1, \dots, \vec{v}_r\}$. If b_{ij} is the multiplicity of the ray spanned by \vec{v}_i in $\text{Trop}(Z_j)$ and m_i is the multiplicity of Z_i in $\text{in}_w(X)$ then the balancing condition implies that $b_{1j}\vec{v}_1 + \dots + b_{rj}\vec{v}_r = 0$, and we also have $m_1b_{i1} + \dots + m_sb_{is} = a_i$, since $\text{Trop}(\text{in}_w(X)) = \text{Star}_w(\text{Trop}(X))$ by [Spe07, Proposition 10.1].

Theorem 6.25. *Let $w \in \text{Trop}(X) \cap N_G$. If $\text{Trop}(X)$ is trivalent at w and one of the edges adjacent to w has multiplicity one, then w has multiplicity one.*

Proof. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the primitive generators of the edge directions in $\text{Trop}(X)$ at w , and let a_i be the multiplicity of the edge in direction \vec{v}_i . The linear span $\langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle$ is two dimensional, since the \vec{v}_i are distinct and satisfy the balancing condition, so any relation among them is a scalar multiple of the relation $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$.

Let Z be an irreducible component of $\text{in}_w(X)$, and let b_i be the multiplicity of the ray spanned by \vec{v}_i in $\text{Trop}(Z)$. Then b_i is a nonnegative integer bounded above by a_i and $b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 = 0$. This relation must be a scalar multiple of the relations given by the a_i , so there is a positive rational number $\lambda \leq 1$ such that $b_i = \lambda a_i$ for all i . If some a_i is one, then λ must also be one and $b_i = a_i$ for all i . Since a_i is the sum of the multiplicities of the ray spanned by \vec{v}_i in the tropicalizations of the components of $\text{in}_w(X)$, it follows that $\text{in}_w(X)$ has no other components, and w has multiplicity one. \blacksquare

Remark 6.26. Initial degenerations at interior points of an edge of multiplicity 1 are always smooth, since they are isomorphic to \mathbf{G}_m by Theorem 4.29.

Remark 6.27. There are other natural combinatorial conditions which can guarantee multiplicity one at a point $w \in N_G$ of $\text{Trop}(X)$ or smoothness of the corresponding initial degeneration. In the case of curves, for example, the argument above works more generally if $\text{Trop}(X)$ is r -valent, the

linear span of the edge directions at w has dimension $r - 1$, and the multiplicities of the edges at w have no nontrivial common factor.

Combining the previous two results and the discussion of tropical hypersurfaces in (2.1), we obtain the following. In order to state the result, we define a *bridge* in a graph Γ to be a segment e such that removing the interior of e disconnects Γ . Note that a bridgeless graph Γ has no vertices of valence one.

Corollary 6.28. *Suppose that $g(\widehat{X}) \geq 1$ and let Σ be the minimal skeleton of \widehat{X}^{an} .*

- (1) *If all vertices of $\text{Trop}(X)$ are trivalent, all edges of $\text{Trop}(X)$ have multiplicity 1, Σ is bridgeless, and $\dim H_1(\Sigma, \mathbf{R}) = \dim H_1(\text{Trop}(X), \mathbf{R})$, then $\text{trop} : \Sigma \rightarrow \text{Trop}(X)$ is an isometry onto its image.*
- (2) *If $X \subset \mathbf{G}_m^2$ is defined by a polynomial $f \in K[x, y]$ whose Newton complex (see Remark 2.2) is a unimodular triangulation, and if $\text{Trop}(X)$ contains a bridgeless connected subgraph Σ' such that $\dim H_1(\Sigma', \mathbf{R}) = g(\widehat{X})$, then \widehat{X} has totally degenerate reduction and $\text{trop} : \Sigma \rightarrow \text{Trop}(X)$ induces an isometry from Σ to Σ' .*

Proof. Let $\Sigma' = \text{trop}(\Sigma)$, which is a compact connected subgraph of $\text{Trop}(X)$. We claim that Σ' is bridgeless. To see this, suppose for the sake of contradiction that Σ' contains a bridge e' . Shrinking e' if necessary, we may assume without loss of generality (since $\text{trop} : \Sigma \rightarrow \Sigma'$ is piecewise linear and surjective) that $\text{trop}^{-1}(e') = \{e_1, \dots, e_r\}$ with $r \geq 1$ and e_1, \dots, e_r disjoint embedded segments in $\mathbf{H}(X^{\text{an}})$. By Corollary 6.11 we must have $r = 1$, i.e., there is a *unique* segment e in X^{an} mapping homeomorphically onto e' . It follows easily that e is a bridge of Σ , contradicting our assumption that Σ is bridgeless. Thus Σ' is bridgeless as claimed.

Now let $g' = \dim H_1(\Sigma, \mathbf{R})$. By Theorem 5.49, Σ is the *unique* bridgeless subgraph of X^{an} whose first homology has dimension (greater than or equal to) g' . By Theorem 6.25, all $w \in \text{Trop}(X) \cap N_G$ have multiplicity 1, so according to Theorem 6.24, there is a (unique) finite subgraph $\widetilde{\Sigma}$ of X^{an} mapping homeomorphically onto Σ' . As Σ' is bridgeless and $\dim H_1(\Sigma', \mathbf{R}) = g'$, the same properties are true of $\widetilde{\Sigma}$. By the uniqueness of Σ , we must have $\Sigma = \widetilde{\Sigma}$. It then follows from Theorem 6.24 that $\text{trop} : \Sigma \rightarrow \text{Trop}(X)$ is an isometry onto its image, proving (1).

We now prove (2). By (2.1), $\text{Trop}(X)$ is trivalent with all edges of multiplicity one. By Theorem 6.24, there is a (unique) finite subgraph Γ of X^{an} mapping homeomorphically onto Σ' via trop . Since Γ is bridgeless and $\dim H_1(\Gamma, \mathbf{R}) = g(\widehat{X})$, it follows that the minimal skeleton Σ of X^{an} is equal to Γ . By the genus formula (5.45.1), \widehat{X} has totally degenerate reduction. By Theorem 6.24, $\text{trop} : \Sigma \rightarrow \text{Trop}(X)$ is an isometry and Σ is isometric to Σ' . \blacksquare

Corollary 6.28 will be important for applications to Tate curves in §7.

Example 6.29. As an example where the hypotheses of Corollary 6.28 are satisfied, consider the genus three curve $X = V(f)$ with

$$f = t^4(x^4 + y^4 + z^4) + t^2(x^3y + x^3z + xy^3 + xz^3 + y^3z + yz^3) + t(x^2y^2 + x^2z^2 + y^2z^2) + x^2yz + xy^2z + xyz^2.$$

See Figure 7.

Example 6.30. The following example shows that it is possible, even under the hypotheses of Theorem 6.24, for the tropicalization map to fail to be faithful. Let K be the completion of $\mathbf{C}\{\{t\}\}$, let $\widehat{X} \subset \mathbf{P}^2$ be defined by the equation

$$(y - 1)^2 = (x - 1)^2(y + 1) + t \cdot xy$$

over $\mathbf{C}\{\{t\}\}$, and let $X = \widehat{X} \cap \mathbf{G}_{m,k}^2$. The above equation degenerates to a nodal rational curve when $t = 0$, the node being $[1 : 1 : 0]$, so it defines a (not strongly) semistable algebraic integral model \mathcal{X} of \widehat{X} (see Remark 5.30(2)) — in fact, \mathcal{X} is a minimal stable model for \widehat{X} , and the associated semistable vertex set only contains one point, so it is minimal as well (see Definition 5.47). Therefore \widehat{X} is an elliptic curve with bad reduction and $\text{in}_0(X) \cong \overline{\mathcal{X}} \cap \mathbf{G}_{m,k}^2$ is reduced and irreducible. The set of

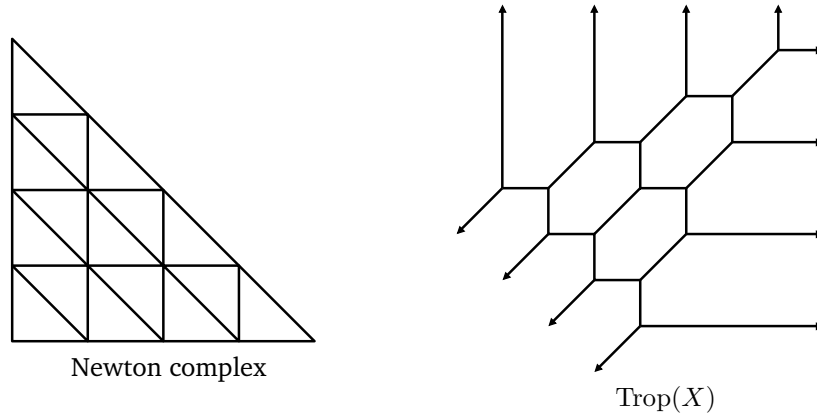


FIGURE 7. The Newton complex and tropicalization of the curve X defined by the polynomial f from Example 6.29. The tropicalization faithfully represents the minimal skeleton of X^{an} .

punctures

$$D := \widehat{X}(K) \setminus X(K) = \{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0], [0 : 3 : 1], [2 : 0 : 1]\}$$

reduce to distinct smooth points of $\overline{\mathcal{X}}$, so if Σ is the minimal skeleton of \widehat{X}^{an} and $\tau_{\Sigma} : \widehat{X}^{\text{an}} \rightarrow \Sigma$ is the retraction, then $\tau_{\Sigma}(x)$ reduces to the generic point of $\overline{\mathcal{X}}$ for all $x \in D$. Therefore the minimal skeleton Γ of X and the tropicalization of X are as shown in Figure 8. We see that $\text{Trop}(X)$ is contractible and everywhere multiplicity one but image of the section does not contain the loop in X^{an} (the loop is contracted to the origin). In particular, trop is not faithful despite the fact that all points in $\text{Trop}(X)$ have multiplicity one.

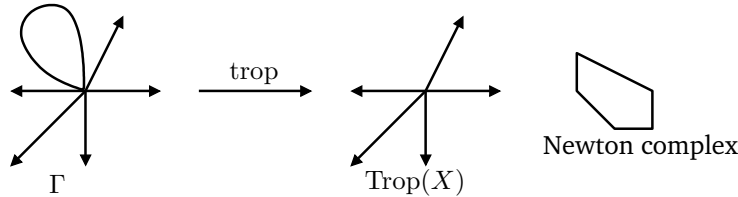


FIGURE 8. The skeleton, tropicalization, and Newton complex of the curve X from Example 6.30. One sees from the Newton complex that the initial degenerations are all multiplicity one away from 0, and $\text{in}_0(X)$ is a rational nodal curve. However the tropicalization crushes the loop in X^{an} to the origin.

7. ELLIPTIC CURVES

Let \widehat{E}/K be an elliptic curve. If \widehat{E} has good reduction then the minimal skeleton Σ of \widehat{E}^{an} is a point, while if \widehat{E} has multiplicative reduction then the minimal skeleton Σ of \widehat{E}^{an} is homeomorphic to a circle of length $-\text{val}(j_{\widehat{E}}) = \text{val}(q_{\widehat{E}})$, where $\widehat{E}^{\text{an}} \cong \mathbf{G}_m^{\text{an}}/q_{\widehat{E}}^{\mathbf{Z}}$ is the Tate uniformization of \widehat{E} (see Remark 5.51). In this section, we use our results on nonarchimedean analytic curves and their tropicalizations to prove some new results (and reinterpret some old results) about tropicalizations of elliptic curves.

7.1. Faithful tropicalization of elliptic curves. As noted in [KMM08, KMM09], a curve in \mathbf{G}_m^2 given by a Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$ cannot have a cycle in its tropicalization, because the Newton complex of a Weierstrass equation does not have an interior vertex. Thus Weierstrass equations are always ‘bad’ from the point of view of tropical geometry. On the other hand, the following result shows that there do always exist ‘good’ plane embeddings of elliptic curves with multiplicative reduction.

Theorem 7.2. *Let \widehat{E}/K be an elliptic curve with multiplicative reduction. Then there is a closed embedding of \widehat{E} in \mathbf{P}^2 , given by a projective plane equation of the form $ax^2y + bxy^2 + cxyz = dz^3$, such that (letting E be the open affine subset of \widehat{E} mapping into the torus \mathbf{G}_m^2) $\text{Trop}(E)$ is a trivalent graph, every point of $\text{Trop}(E)$ has a smooth and irreducible initial degeneration (hence tropical multiplicity 1), and the minimal skeleton of \widehat{E} is faithfully represented by the tropicalization map. In particular, $\text{Trop}(E)$ contains a cycle of length $-\text{val}(j_{\widehat{E}})$.*

Proof. Let $q = q_{\widehat{E}}$ be the Tate parameter, so that $\widehat{E}^{\text{an}} \cong \mathbf{G}_m^{\text{an}}/q^{\mathbf{Z}}$. Choose a cube root $q^{1/3} \in K^{\times}$ of q and let $\alpha, \beta \in \widehat{E}(K)$ correspond under the Tate isomorphism to the classes of $q^{1/3}$ and $(q^{1/3})^2$, respectively. Recall that a divisor $D = \sum a_P(P)$ on an elliptic curve is principal if and only if $\sum a_P = 0$ and $\sum a_PP = 0$ in the group law on the curve. In particular, there exist rational functions f and g on \widehat{E} (unique up to multiplication by a nonzero constant) such that $\text{div}(f) = 2(\alpha) - (\beta) - (0)$ and $\text{div}(g) = 2(\beta) - (\alpha) - (0)$. Let $\psi : \widehat{E} \rightarrow \mathbf{P}^2$ be the morphism associated to the rational map $[f : g : 1]$. Since $1, f, g$ form a basis for $L(D)$ with $D = (\alpha) + (\beta) + (0)$ and D is very ample [Har77, Corollary IV.3.2(b)], ψ is a closed immersion.

Let Γ be the minimal skeleton of E , i.e., the smallest closed connected subset of \widehat{E}^{an} containing the skeleton Σ of \widehat{E}^{an} and the three points $\alpha, \beta, 0$, with those three points removed. Recall that Σ is isometric to a circle of circumference $\ell_{\text{an}}(\Sigma) = -\text{val}(j_{\widehat{E}}) = \text{val}(q)$. The natural map

$$K^{\times} \twoheadrightarrow \widehat{E}(K) \hookrightarrow \widehat{E}^{\text{an}} \twoheadrightarrow \Sigma \xrightarrow{\sim} \mathbf{R}/\ell\mathbf{Z}$$

is given by $z \mapsto [\text{val}(z)]$; in particular, if $\tau_{\Sigma} : \widehat{E}^{\text{an}} \twoheadrightarrow \Sigma$ denotes the canonical retraction, we have $\tau_{\Sigma}(0) = [0]$, $\tau_{\Sigma}(\alpha) = [\frac{1}{3}\text{val}(q)]$, and $\tau_{\Sigma}(\beta) = [\frac{2}{3}\text{val}(q)]$. Thus Γ is a circle with an infinite ray emanating from each of three equally spaced points $O = \tau_{\Sigma}(0)$, $A = \tau_{\Sigma}(\alpha)$, $B = \tau_{\Sigma}(\beta)$ along the circle (see Figure 9 below). The tropicalization map $\text{trop} : E = \widehat{E}^{\text{an}} \setminus \{0, \alpha, \beta\} \rightarrow \mathbf{R}^2$ corresponding to the embedding $E \hookrightarrow \mathbf{G}_m^2$ given by (f, g) factors through the retraction onto Γ .

The map $\text{trop} : \Gamma \rightarrow \mathbf{R}^2$ can be determined (up to an additive translation) using the Slope Formula (Theorem 5.69; also see Proposition 5.10) by solving an elementary graph potential problem. The result is as follows. The function $\text{val}(f) = -\log|f|$ has slope -1 along the ray from O to 0 , slope 2 along the ray from A to α , and slope -1 along the ray from B to β . On Σ , it has slope 1 along the segment from O to A , slope -1 along the segment from A to B , and slope 0 along the segment from B to O . Similarly, the function $\text{val}(g) = -\log|g|$ has slope -1 along the ray from O to 0 , slope -1 along the ray from A to α , and slope 2 along the ray from B to β . On Σ , it has slope 0 along the segment from O to A , slope 1 along the segment from A to B , and slope -1 along the segment from B to O . Thus (up to a translation on \mathbf{R}^2) $\text{Trop}(E)$ is a trivalent graph consisting of a triangle with an infinite ray emanating from each of the vertices as in Figure 9.

Since the expansion factor along every edge of Γ is equal to 1 by (6.6.1), it follows from Corollary 6.9 that the tropical multiplicity of every edge of $\text{Trop}(E)$ is 1. By Theorem 6.25, the multiplicity at every vertex of $\text{Trop}(E)$ is 1 as well, and in fact the initial degenerations are smooth and irreducible since the Newton complex is unimodular (see Figure 9). Since the expansion factor is 1 along every edge of Σ and $\text{trop}|_{\Sigma}$ is a homeomorphism, it follows that Σ is faithfully represented. The bounded edges of $\text{Trop}(E)$ form a triangle each of whose sides has lattice length $\text{val}(q)/3$.

The only thing which remains to be proved is that $\psi(\widehat{E}) \subset \mathbf{P}^2$ is cut out by an equation of the form indicated in the statement of the theorem. This follows from the Riemann-Roch theorem: the

functions $1, fg, f^2g, fg^2$ all belong to the 3-dimensional vector space $L(3(0))$ and hence there is a nonzero linear relation between them. (This argument is similar to [Har77, Proposition IV.4.6].) ■

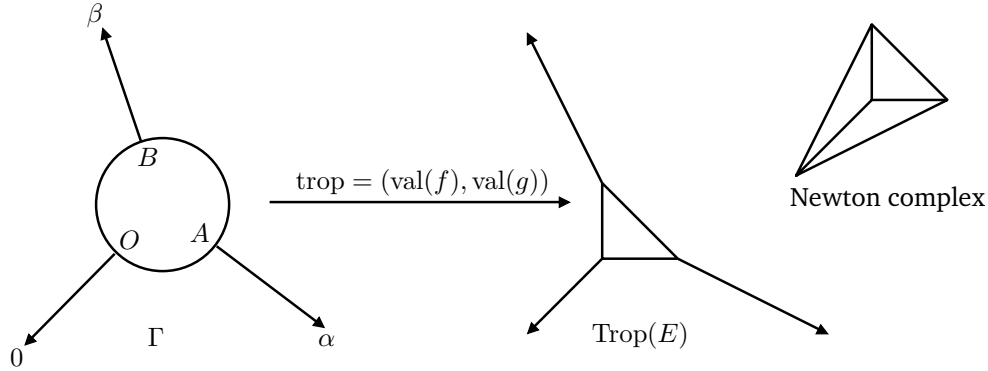


FIGURE 9. The skeleton Γ of E , the tropicalization of E , and the Newton complex of the equation $ax^2y + bxy^2 + cxyz = dz^3$ defining E , where E is as in the proof of Theorem 7.2. The minimal skeleton Σ of \widehat{E} is the circle contained in Γ . The tropicalization faithfully represents Σ , so $\ell_{\text{Trop}}(\text{Trop}(\Sigma)) = \ell_{\text{an}}(\Sigma)$.

We can also use our theorems to give more conceptual proofs of many of the results from [KMM08, KMM09]. For example, we have the following theorem which was proved in [KMM09] by a brute-force computation:

Theorem 7.3. *Let $E \subset \mathbf{G}_m^2$ be the intersection of an elliptic curve $\widehat{E} \subset \mathbf{P}^2$ with \mathbf{G}_m^2 . Assume that (i) $\text{Trop}(E)$ contains a cycle C , (ii) all edges of $\text{Trop}(E)$ have multiplicity 1, and (iii) $\text{Trop}(E)$ is trivalent. Then $\ell_{\text{Trop}}(C) = -\text{val}(j_{\widehat{E}})$.*

Proof. This follows immediately from Corollary 6.28(1). ■

Remark 7.4. Conditions (i)–(iii) from Theorem 7.3 are automatically satisfied if the Newton complex of the defining polynomial for E is a unimodular triangulation with a vertex lying in the interior of the Newton polygon, as in Figure 9. Varying the valuation of the coefficient corresponding the interior vertex while keeping all other coefficients fixed gives a natural map from an annulus in \mathbf{G}_m to the j -line, which is finite and flat onto an annulus in the j -line, by Proposition 5.2(2). In particular, given a tropical plane curve dual to such a Newton complex and an elliptic curve E with j -invariant equal to minus the length of the loop, there is an embedding of E into a toric variety such that the tropicalization of the intersection with \mathbf{G}_m^2 is faithful and equal to the given tropical curve. See [CS12] for explicit constructions of such embeddings for tropical curves of “honeycomb normal form,” including an algorithm for finding the honeycomb form of an elliptic curve, parameterization by theta functions, the tropicalization of the inflection points, and relations to the group law.

Remark 7.5. A different (but related) conceptual explanation for Theorem 7.3 is given in [HK11, Proposition 7.7].

Let us say that a closed embedding of an elliptic curve \widehat{E}/K in some toric variety is *certifiably of genus 1* if $\text{Trop}(E)$ satisfies conditions (i)–(iii) from Theorem 7.3. Note that the cycle C in any such embedding satisfies $\ell_{\text{Trop}}(C) = -\text{val}(j_{\widehat{E}})$, by Theorem 7.3. Combining Theorems 7.2 and 7.3, we obtain:

Corollary 7.6. *An elliptic curve \widehat{E}/K has multiplicative reduction if and only if it has a closed embedding in \mathbf{P}^2 which is certifiably of genus 1.*

7.7. Non-faithful tropicalization of elliptic curves. As a counterpart to Theorem 7.2, we have the following result showing that even when $\text{Trop}(E)$ contains a cycle, the cycle might have the ‘wrong’

length. As in the statement of Theorem 7.2, if \widehat{E} is an elliptic curve embedded in a toric variety with dense torus \mathbf{T} then we let $E = \widehat{E} \cap \mathbf{T}$.

Theorem 7.8. *Let \widehat{E}/K be an elliptic curve with multiplicative reduction and let $\ell = -\text{val}(j_{\widehat{E}}) = \text{val}(q_{\widehat{E}}) > 0$. Let Σ be the minimal skeleton of \widehat{E} .*

- (1) *There is a closed embedding of \widehat{E} in \mathbf{P}^2 whose tropicalization contracts Σ to a point.*
- (2) *There is a closed embedding of \widehat{E} in \mathbf{P}^2 such that $\text{Trop}(E)$ contains a cycle of lattice length $\frac{3}{4}\ell < \ell$.*
- (3) *There is a closed embedding of \widehat{E} in $\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ such that $\text{Trop}(E)$ contains a cycle of lattice length $\frac{5}{4}\ell > \ell$.*
- (4) *There is a closed embedding of \widehat{E} in $\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ such that $\text{Trop}(E)$ contains a cycle of lattice length ℓ but Σ is not faithfully represented.*

Proof. The constructions are similar to the proof of Theorem 7.2.

For (1), assume $\text{char}(K) \neq 3$, choose a primitive cube root of unity ω in K^\times , and let $P, Q \in \widehat{E}(K)$ correspond to $-1, \omega \in K^\times$, respectively. We denote the identity element of $\widehat{E}(K)$ by 0. There are rational functions x', y' on \widehat{E} such that $\text{div}(x') = 2(P) - 2(0)$ and $\text{div}(y') = 3(Q) - 3(0)$. Since $1, x', y'$ form a basis for $L(3(0))$ and $3(0)$ is very ample, the morphism $\iota : \widehat{E} \rightarrow \mathbf{P}^2$ associated to the rational map $[x' : y' : 1]$ is a closed immersion. By the Slope Formula (Theorem 5.69), the tropicalization of this embedding is constant on the skeleton Σ of \widehat{E} since the retractions of $\text{div}(x')$ and $\text{div}(y')$ to Σ are both 0. We leave the case $\text{char}(K) = 3$ as an exercise for the reader.

For (2), choose a fourth root $q^{1/4}$ of q in K and let $\alpha, \beta, \gamma \in \widehat{E}(K)$ correspond to $q^{1/4}, (q^{1/4})^2$, and $(q^{1/4})^3$, respectively. Let Γ be the minimal skeleton of E , so Γ is a circle with an infinite ray emanating from each of four equally spaced points $O = \tau_\Sigma(0), A = \tau_\Sigma(\alpha), B = \tau_\Sigma(\beta), C = \tau_\Sigma(\gamma)$ along the circle (see Figure 10). There are rational functions f_1 and f_2 on \widehat{E} such that $\text{div}(f_1) = (\alpha) + (\beta) - (\gamma) - (0)$ and $\text{div}(f_2) = 3(\beta) - 2(\gamma) - (0)$. Let $\psi : \widehat{E} \rightarrow \mathbf{P}^2$ be the morphism associated to the rational map $[f_1 : f_2 : 1]$. Since $1, f_1, f_2$ form a basis for $L(D)$ with $D = 2(\gamma) + (0)$ and D is very ample, ψ is a closed immersion.

By the Slope Formula (Theorem 5.69; also see Proposition 5.10), $\text{Trop}(E)$ is a triangle $T = \text{trop}(\Sigma)$ with an infinite ray emanating from each of the vertices $\text{trop}(A)$ and $\text{trop}(B)$ and two infinite rays emanating from the vertex $\text{trop}(O) = \text{trop}(C)$ as in Figure 10. The expansion factor along the segment from C to O is zero and all other expansion factors along Γ are 1. It follows that the lattice length of T is $\frac{3}{4}\ell$, which proves (2). (Note that all edges in $\text{Trop}(E)$ have multiplicity 1; the expansion factor is 1 by (6.6.1) so this follows from Corollary 6.9. The multiplicity of the vertex $w_0 = \text{trop}(O) = \text{trop}(C)$ of $\text{Trop}(E)$ must be strictly bigger than 1 since otherwise, by Theorem 6.24, the map $\text{trop} : \Gamma \rightarrow \text{Trop}(E)$ would have a section in some neighborhood of w_0 , which is visibly not the case.)

For (3), choose f_3 so that $\text{div}(f_3) = 3(\beta) + (\gamma) - (\alpha) - 3(0)$. On Σ , $-\log|f_3|$ has slope 1 along the segment from O to A , slope 2 along the segment from A to B , slope -1 along the segment from B to C , and slope -2 along the segment from C to O . Let $\varphi_3 : \widehat{E} \rightarrow \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ be the map given in affine coordinates by (x', y', f_2, f_3) . Since the map $\widehat{E} \rightarrow \mathbf{P}^2$ given by (x', y') is a closed immersion, φ_3 is also a closed immersion. The tropicalization map $\text{trop}(\varphi_3)$ maps Σ onto a planar convex quadrilateral of total length $5\ell/4$. The edge of Σ from C to O maps with an expansion factor of 2, while the other three edges of Σ are each mapped isometrically onto their images.

For (4), choose f_4 so that $\text{div}(f_4) = 7(\beta) - 2(\alpha) - 4(\gamma) - (0)$. On Σ , $-\log|f_4|$ has slope 1 along the segment from O to A , slope 3 along the segment from A to B , slope -4 along the segment from B to C , and slope 0 along the segment from C to O . Let $\varphi_4 : \widehat{E} \rightarrow \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ be the closed immersion given in affine coordinates by (x', y', f_2, f_4) . The tropicalization map $\text{trop}(\varphi_4)$ maps Σ onto a triangle of length ℓ , but this agreement of lengths happens ‘by accident’. In fact, what is happening is that the tropicalization map collapses the edge of Σ from C to O while expanding the edge from B to C by a factor of 2; the other two edges map isometrically onto their images. Thus $\text{Trop}(\Sigma)$ is a triangle of

total length ℓ with one side of length $\ell/2$ and multiplicity 2 and two sides of length $\ell/4$ and multiplicity 1. \blacksquare

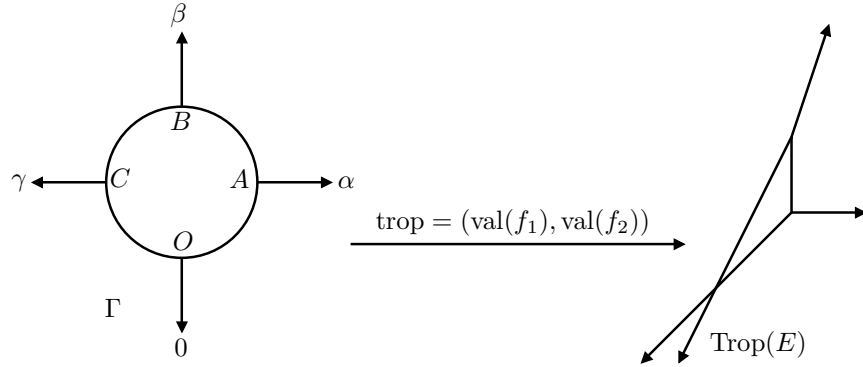


FIGURE 10. The skeleton Γ of E and the tropicalization of E , where E is as in the proof of Theorem 7.8(2). In this case $\ell(\text{Trop}(\Sigma)) = \frac{3}{4}\ell(\Sigma)$. The segment from C to O is collapsed to a point by trop , accounting for the factor of $\frac{3}{4}$.

7.9. Speyer's well-spacedness condition. In this section we explain how Speyer's well-spacedness condition [Spe07] can be interpreted and generalized as a statement about the analytification of an elliptic curve \widehat{E}/K .

Let Σ be the minimal skeleton of \widehat{E} . For $P, Q \in \widehat{E}(K)$, define $i(P, Q) \in \mathbf{R}_{\geq 0} \cup \{\infty\}$ as follows:

$$i(P, Q) = \begin{cases} 0 & \text{if } \tau_{\Sigma}(P) \neq \tau_{\Sigma}(Q) \\ \text{dist}(P \vee Q, \Sigma) & \text{if } \tau_{\Sigma}(P) = \tau_{\Sigma}(Q) \end{cases}$$

where $\tau_{\Sigma} : \widehat{E}^{\text{an}} \rightarrow \Sigma$ is the retraction map, $P \vee Q$ is the first point where the geodesic paths from P to Σ and Q to Σ meet, and $\text{dist}(x, \Sigma)$ is the distance (in the natural metric on $\mathbf{H}(\widehat{E}^{\text{an}})$) from $x \in \widehat{E}^{\text{an}}$ to its retraction $\tau_{\Sigma}(x) \in \Sigma$. By convention we set $i(P, P) = +\infty$. Since translation by a point $P \in \widehat{E}(K)$ is an automorphism of \widehat{E} , it induces an isometry on $\mathbf{H}(\widehat{E}^{\text{an}})$; therefore $i(P, Q)$ only depends on the difference $P - Q$ in $\widehat{E}(K)$, i.e., $i(P, Q) = \iota(P - Q)$ with $\iota(R) = i(R, 0)$.

The following lemma shows that $\|P, Q\| := \exp(-i(P, Q))$ is an *ultrametric* on $\widehat{E}(K)$:

Lemma 7.10.

- (1) For any points $P, Q, R \in \widehat{E}(K)$ we have $i(P, Q) \geq \min\{i(P, R), i(Q, R)\}$, with equality if $i(P, R) \neq i(Q, R)$.
- (2) If $m \in \mathbf{Z}$ is an integer such that $|m| = 1$ in K then $i(mP, mQ) = i(P, Q)$ for any $P, Q \in \widehat{E}(K)$ such that $i(P, Q) > 0$.

Proof. We begin by proving (1). If either $i(P, R) = 0$ or $i(Q, R) = 0$ then the inequality is trivial. Moreover, by translation invariance of i we may assume that $R = 0$. So we are reduced to showing that if $\iota(P) > 0$ and $\iota(Q) > 0$ then $\iota(P - Q) \geq \min\{\iota(P), \iota(Q)\}$.

Let \mathfrak{E} be the semistable formal model of \widehat{E} corresponding to the semistable vertex set $\{\tau_{\Sigma}(0)\}$ (see Remark 5.51; note that $\Sigma = \Sigma(\widehat{E}, \{\tau_{\Sigma}(0)\})$). Then $\overline{\mathfrak{E}}$ is a nodal rational curve, and the smooth locus $\overline{\mathfrak{E}}^{\text{sm}}$ is a group scheme isomorphic to $\mathbf{G}_{m,k}$. The subset $\widehat{E}^1(K) := \{P \in \widehat{E}(K) : \iota(P) > 0\}$ is the formal fiber over the identity element of $\overline{\mathfrak{E}}^{\text{sm}}$, hence is a subgroup; in fact, $\widehat{E}^1(K)$ is isomorphic to the group $\mathfrak{m} = \{z \in K : |z| < 1\}$ with the law of composition given by a one-parameter formal group law F over R [Sil09, Proposition VII.2.1], and the restriction of ι to $\widehat{E}^1(K)$ corresponds to the valuation on \mathfrak{m} under this identification. The desired inequality follows since a group law on \mathfrak{m} given by a power series with coefficients in R is obviously ultrametric.

In the situation of (2), as above we are reduced to showing that $\iota(mP) = \iota(P)$ when $\iota(P) > 0$. This is true because m is the coefficient of the linear term of the power series for multiplication by m under F [Sil09, Proposition IV.2.3], and all other terms have larger valuation. ■

Since $\iota(P) = \iota(-P)$, an equivalent formulation of Lemma 7.10(1) is that for any $P, Q \in \widehat{E}(K)$ we have $\iota(P + Q) \geq \min\{\iota(P), \iota(Q)\}$, with strict inequality if $\iota(P) \neq \iota(Q)$.

If f is a nonconstant rational function on \widehat{E} , define N_f to be the set of all $x \in \mathbf{H}(\widehat{E}^{\text{an}})$ such that $\log|f|$ is non-constant in every open neighborhood of x . Equivalently, for $x \in \mathbf{H}(\widehat{E}^{\text{an}})$ let $T_x(f)$ be the (finite) set of tangent directions at x along which the derivative of $\log|f|$ is nonzero. Then N_f is the set of all $x \in \widehat{E}^{\text{an}}$ such that $T_x(f) \neq \emptyset$. By Theorem 5.69(1,2), N_f is a union of finitely many edges of the minimal skeleton Γ_f of the curve obtained from \widehat{E}^{an} by removing all zeros and poles of f .

Theorem 7.11. *Suppose that K has residue characteristic zero. Let f be a nonconstant rational function on \widehat{E} and assume that there exists $x \in N_f$ such that $\text{dist}(x, \Sigma) < \text{dist}(y, \Sigma)$ for all $y \in N_f$ with $y \neq x$. Assume also that $\Sigma \cap N_f = \emptyset$. Then $|T_x(f)| \geq 3$.*

In other words, either the minimum distance from N_f to the skeleton is achieved at two distinct points, or else the minimum is achieved at a unique point at which $\log|f|$ has nonzero slope in at least three different tangent directions.

Proof. By the Slope Formula (Theorem 5.69), the sum of the outgoing slopes of $\log|f|$ at x is zero, so $|T_x(f)| \geq 2$. Assume for the sake of contradiction that $|T_x(f)| = 2$ and write $T_x(f) = \{v, v'\}$. Our hypotheses imply that $x \notin \Sigma$ and that Σ lies in a single connected component of $\widehat{E}^{\text{an}} \setminus \{x\}$.

Let $B(x, v)$ (resp. $B(x, v')$) be the open set consisting of all $z \in \widehat{E}^{\text{an}}$ lying in the tangent direction v (resp. v'), so that $B(x, v)$ and $B(x, v')$ are connected components of $\widehat{E}^{\text{an}} \setminus \{x\}$ which are disjoint from Σ . Let D_v be the restriction of $\text{div}(f)$ to $B(x, v)$ and let $D_{v'}$ be the restriction of $\text{div}(f)$ to $B(x, v')$. By the Slope Formula, we have $m = \deg(D_v) = -\deg(D_{v'})$ for some nonzero integer m . Without loss of generality, we may assume that $m > 0$. Let $\delta = \text{dist}(x, \Sigma) > 0$.

We claim that $\text{div}(f)$ can be written as

$$(7.11.1) \quad \text{div}(f) = m((P) - (Q)) + \sum_j ((A_j) - (B_j))$$

with $i(P, Q) = \delta$ and $i(A_j, B_j) > \delta$ for all j . Because we have assumed that K has residue characteristic zero, we have $|m| = 1$, and therefore $i(m(P), m(Q)) = i(P, Q) = \delta$ by Lemma 7.10(2). Since $mP - mQ = \sum (B_j - A_j)$ in the group law on $\widehat{E}(K)$, we obtain a contradiction to Lemma 7.10(1).

To prove the claim, we use a trick due to D. Speyer. Suppose $D_v = (P_1) + \dots + (P_r) - (Q_1) - \dots - (Q_s)$ and $D_{v'} = (P'_1) + \dots + (P'_{r'}) - (Q'_1) - \dots - (Q'_{s'})$ with $r - s = m$ and $s' - r' = m$. Then

$$\begin{aligned} D_v + D_{v'} &= m((P_1) - (Q'_1)) + \sum_{j=1}^s ((P_j) - (Q_j)) + \sum_{j=s+1}^r ((P_j) - (P_1)) \\ &\quad + \sum_{j=1}^{r'} ((P'_j) - (Q'_j)) + \sum_{j=r'+1}^{s'} ((Q'_1) - (Q'_j)). \end{aligned}$$

Note that $i(P_1, Q'_1) = \delta$ but that $i(P_j, Q_j) > \delta$ for all $j = 1, \dots, s$, $i(P_j, P_1) > \delta$ for all $j = s+1, \dots, r$, $i(P'_j, Q'_j) > \delta$ for all $j = 1, \dots, r'$, and $i(Q'_1, Q'_j) > \delta$ for all $j = r'+1, \dots, s'$.

Let C_1, \dots, C_t be the connected components of N_f , labeled so that C_1 is the component containing x . By the Slope Formula (Theorem 5.69), for each j the restriction D_j of $\text{div}(f)$ to C_j is a nonzero divisor of degree zero and $\text{div}(f) = \sum_j D_j$. Moreover, if $A, B \in \widehat{E}(K) \cap C_j$ then the unique geodesic paths from A to Σ and B to Σ must pass through the unique point x_j of C_j closest to Σ , so for $j \geq 2$ we have $i(A, B) > \delta$. The claim now follows since $D_1 = D_v + D_{v'}$ can be written as above, and by what we have just said we can write each D_j for $j \geq 2$ as a sum of divisors of the form $(A) - (B)$ with $i(A, B) > \delta$. ■

In particular, we obtain the necessity of Speyer's well-spacedness condition for a genus 1 tropical curve to lift:

Corollary 7.12. (Speyer) *Suppose that K has residue characteristic zero. Let E be a dense open subset of an elliptic curve \widehat{E} over K with multiplicative reduction and let $\psi : E \hookrightarrow \mathbf{T}$ be a closed embedding of E in a torus \mathbf{T} . Assume that (i) every vertex of $\text{Trop}(E)$ is trivalent, (ii) every edge of $\text{Trop}(E)$ has multiplicity one, and (iii) $\text{Trop}(E)$ contains a cycle Σ' which is contained in a hyperplane H . If W_H denotes the closure in $N_{\mathbf{R}}$ of the set of points of $\text{Trop}(E)$ not lying in H , then there is no single point of W_H which is closest to Σ' .*

In other words, ‘the minimum distance from points of $\text{Trop}(E)$ not lying in H to the cycle must be achieved twice’.

Proof. We may assume that $\psi : E \rightarrow \mathbf{T} \cong \mathbf{G}_m^n$ is given by (f_1, \dots, f_n) with $\log |f_n|$ equal to a constant c on Σ and that H is the hyperplane $x_n = c$. Let Γ be the minimal skeleton of E . By Corollary 6.28(1), we see that $\text{trop} : \Gamma \rightarrow \text{Trop}(E)$ is an isometry. Since $N_{f_n} \subset \Gamma$, the result now follows from Theorem 7.11. \blacksquare

Remark 7.13. E. Katz [Kat10] and T. Nishinou [Nis10] have recently obtained other kinds of generalizations of Speyer's well-spacedness condition. Their generalized conditions apply to curves of higher genus.

8. TROPICAL ELIMINATION THEORY AND TROPICAL IMPLICITIZATION

8.1. A generalization of the Sturmfels-Tevelev multiplicity formula. Using the algebraic techniques from this paper, we are able to generalize the Sturmfels-Tevelev multiplicity formula [ST08, Theorem 1.1], which is an important tool in tropical implicitization, to the non-constant coefficient case (and also to non-smooth points). An algebraic proof is given in an appendix to [OP10].

Let $X \subset \mathbf{T}$ be a closed subvariety, i.e., a reduced and irreducible closed subscheme. Let $\alpha : \mathbf{T} \rightarrow \mathbf{T}'$ be a homomorphism of tori that induces a generically finite map of degree δ from X to X' , where X' is the closure of $\alpha(X)$. Then, set theoretically, $\text{Trop}(X')$ is the image of $\text{Trop}(X)$ under the induced linear map $A : N_{\mathbf{R}} \rightarrow N'_{\mathbf{R}}$ [Tev07, Proposition 3]. The fundamental problem of tropical elimination theory is to determine the multiplicities on the maximal faces of $\text{Trop}(X')$ from those on the maximal faces of $\text{Trop}(X)$.

Theorem 8.2. *Let $\alpha : \mathbf{T} \rightarrow \mathbf{T}'$ be a homomorphism of algebraic tori over K and let X be a closed subvariety of \mathbf{T} . Let X' be the schematic image of X in \mathbf{T}' , let $f : X \rightarrow X'$ be the restriction of α to X , and let $F = \text{trop}(f) : \text{Trop}(X) \rightarrow \text{Trop}(X')$ be the restriction of the linear map $A : N_{\mathbf{R}} \rightarrow N'_{\mathbf{R}}$ induced by α . Suppose that f is generically finite of degree δ . Then for any point $w' \in \text{Trop}(X') \cap N'_G$ such that $|F^{-1}(w')| < \infty$, we have*

$$(8.2.1) \quad m_{\text{Trop}}(w') = \frac{1}{\delta} \sum_{w \in F^{-1}(w')} \sum_{\bar{\mathcal{C}} \subset \bar{\mathcal{X}}^w} \text{mult}_{\bar{\mathcal{X}}^w}(\bar{\mathcal{C}}) [\bar{\mathcal{C}} : \text{im}(\bar{\mathcal{C}})],$$

where the second sum runs over all irreducible components $\bar{\mathcal{C}}$ of $\bar{\mathcal{X}}^w$ and where $\text{im}(\bar{\mathcal{C}})$ is the image of $\bar{\mathcal{C}}$ in $(\bar{\mathcal{X}}')^{w'}$.

In order to use the projection formula (3.32), we will need the following lemma.

Lemma 8.3. *Let X, X' be integral finite-type K -schemes and let $f : X \rightarrow X'$ be a generically finite dominant morphism of degree δ . Let $\mathcal{U}' \subset (X')^{\text{an}}$ be an analytic domain and let $\mathcal{U} = (f^{\text{an}})^{-1}(\mathcal{U}')$. If $f^{\text{an}}|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}'$ is finite then it has pure degree δ (3.20).*

Proof. By [Har77, Exercise II.3.7], there is a dense open subscheme $U' \subset X'$ such that $U := f^{-1}(U') \rightarrow U'$ is finite. Shrinking U' if necessary, we assume that U' is smooth. By Proposition 3.31 the morphism $U^{\text{an}} \rightarrow (U')^{\text{an}}$ is pure of degree δ . Let $\mathcal{U}, \mathcal{U}'$ be as in the statement of the Lemma. By Proposition 3.30(2), we may assume that $\mathcal{U} = \mathcal{M}(\mathcal{A})$ and $\mathcal{U}' = \mathcal{M}(\mathcal{A}')$ are affinoid. By [Con99, Lemma A.1.2(2)], \mathcal{U} and \mathcal{U}' are equidimensional of the same dimension as X and X' . Therefore

$\mathcal{U}' \cap (X' \setminus U')^{\text{an}}$ is nowhere dense in \mathcal{U}' . If $\mathcal{V}' = \mathcal{M}(\mathcal{B}')$ is any connected affinoid subdomain of $\mathcal{U}' \cap (U')^{\text{an}}$ then \mathcal{B}' is a domain because \mathcal{V}' is smooth, and $\mathcal{V}' := (f^{\text{an}})^{-1}(\mathcal{V}') \rightarrow \mathcal{V}'$ has (pure) degree δ because $\mathcal{V}' \subset (U')^{\text{an}}$. Since $\mathcal{U}' \cap (X' \setminus U')^{\text{an}}$ is nowhere dense in \mathcal{U}' , we can choose \mathcal{V}' such that $\text{Spec}(\mathcal{B}') \rightarrow \text{Spec}(\mathcal{A}')$ takes the generic point of $\text{Spec}(\mathcal{B}')$ to any given generic point of $\text{Spec}(\mathcal{A}')$. Hence $\mathcal{U} \rightarrow \mathcal{U}'$ has pure degree δ . \blacksquare

Proof of Theorem 8.2. Let $w' \in \text{Trop}(X') \cap N'_G$ be a point with finite preimage under F . Let

$$\mathcal{X}^{w'} = \text{trop}^{-1}(F^{-1}(w')) \cap X^{\text{an}} = \coprod_{w \in F^{-1}(w')} \mathcal{X}^w.$$

This is an affinoid domain in X^{an} because it is a closed subspace of the affinoid $\text{trop}^{-1}(F^{-1}(w')) = \coprod_{w \in F^{-1}(w')} \mathcal{U}^w$. We claim that $\mathcal{X}^{w'} \rightarrow (\mathcal{X}')^{w'}$ is a finite morphism. It suffices to show that the composite $\mathcal{X}^{w'} \rightarrow \mathcal{U}^{w'}$ is a finite morphism, where $\mathcal{U}^{w'} = \text{trop}^{-1}(w') \subset (\mathbf{T}')^{\text{an}}$. This follows exactly as in the proof of Theorem 4.31: since $F^{-1}(w')$ is bounded, there is an affinoid domain of \mathbf{T}^{an} contained in $(\alpha^{\text{an}})^{-1}(\mathcal{U}^{w'})$ and containing $\mathcal{X}^{w'}$ in its relative interior. This means that the morphism $\mathcal{X}^{w'} \rightarrow (\mathcal{X}')^{w'}$ is proper, thus finite because both spaces are affinoid. Hence by Lemma 8.3 the morphism $\mathcal{X}^{w'} \rightarrow (\mathcal{X}')^{w'}$ has pure degree δ . Let $\mathfrak{X}^{w'} := \coprod_{w \in F^{-1}(w')} \mathfrak{X}^w$. The generic fiber of $\mathfrak{X}^{w'}$ is $\mathcal{X}^{w'}$, and the natural morphism $\mathfrak{X}^{w'} \rightarrow (\mathfrak{X}')^{w'}$ is finite by Proposition 3.13(2) and takes generic points to generic points by Proposition 3.15. By the projection formula (3.32) the induced morphism $\overline{\mathfrak{X}}^{w'} \rightarrow (\overline{\mathfrak{X}}')^{w'}$ has pure degree δ , so summing (3.24.1) over all irreducible components $\overline{\mathfrak{C}}'$ of $(\overline{\mathfrak{X}}')^{w'}$ yields

$$\delta \cdot m_{\text{Trop}}(w') = \delta \sum_{\overline{\mathfrak{C}}' \subset (\overline{\mathfrak{X}}')^{w'}} \text{mult}_{(\overline{\mathfrak{X}}')^{w'}}(\overline{\mathfrak{C}}') = \sum_{\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}^{w'}} \text{mult}_{\overline{\mathfrak{X}}^{w'}}(\overline{\mathfrak{C}}) [\overline{\mathfrak{C}} : \text{im}(\overline{\mathfrak{C}})].$$

Since $\mathfrak{X}^{w'} = \coprod_{w \in F^{-1}(w')} \mathfrak{X}^w$, this is the desired multiplicity formula. \blacksquare

As a consequence of Theorem 8.2, we obtain:

Corollary 8.4. *Let $\alpha : \mathbf{T} \rightarrow \mathbf{T}'$ be a homomorphism of algebraic tori over K and let $A = \text{trop}(\alpha) : N_{\mathbf{R}} \rightarrow N'_{\mathbf{R}}$ be the natural linear map. Let X be a closed subvariety of \mathbf{T} , and suppose that α induces a generically finite morphism of degree δ from X onto its schematic image X' in \mathbf{T}' . After subdividing, we may assume that A maps each face of $\text{Trop}(X)$ onto a face of $\text{Trop}(X')$. Let σ' be a maximal face of $\text{Trop}(X')$. Then*

$$m(\sigma') = \frac{1}{\delta} \sum_{A(\sigma) = \sigma'} m(\sigma) \cdot [N'_{\sigma'} : A(N_{\sigma})].$$

(Here N_{σ} and $N'_{\sigma'}$ are the sublattices of N and N' parallel to σ and σ' , respectively.)

Proof. If w' is a smooth point of $\text{Trop}(X')$ and w is a smooth point of $\text{Trop}(X)$ with $A(w) = w'$, then $(\overline{\mathfrak{X}}')^{w'} \cong Y' \times T'(w')$ and $\overline{\mathfrak{X}}^w \cong Y \times T(w)$ with Y, Y' zero-dimensional schemes of length $m_{\text{Trop}}(w)$ and $m_{\text{Trop}}(w')$, respectively and $T(w), T'(w')$ algebraic tori of dimension $\dim(X) = \dim(X')$ (cf. Remark 4.28). Moreover, α induces a finite homomorphism $T(w) \rightarrow T'(w')$ of degree $[N'_{\sigma'} : A(N_{\sigma})]$. In this situation, the quantity $[\overline{\mathfrak{C}} : \text{im}(\overline{\mathfrak{C}})]$ appearing in (8.2.1) is equal to $[T(w) : T'(w')] = [N'_{\sigma'} : A(N_{\sigma})]$, and $m_{\text{Trop}}(w) = \sum_{\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}^w} \text{mult}_{\overline{\mathfrak{X}}^w}(\overline{\mathfrak{C}})$, so we are reduced to Theorem 8.2. \blacksquare

Remark 8.5. The original Sturmfels-Tevelev multiplicity formula is the special case of Corollary 8.4 in which $K = k\{\{T\}\}$ and X is defined over k .

8.6. Using nonarchimedean analytic spaces to draw tropical curves. There are various methods available for plotting tropicalizations of algebraic varieties. For example, if $X \subset \mathbf{G}_m^n$ is a hypersurface then one can use the theory of Newton complexes. More generally, one can use Gröbner theory to determine a ‘tropical basis’ for $\text{Trop}(X)$ and then intersect the corresponding tropical hypersurfaces (see [MS09] for details). In this section we describe a method for drawing tropical curves using nonarchimedean analytic spaces. This method is particularly useful in the case of parametrized rational curves in \mathbf{G}_m^n . The first author’s REU student Melanie Dunn has written a Matlab program to

implement the method in special cases; Example 8.12 below is due to her. The idea is to directly compute the image of the valuation map on $X(K)$; the various cases which arise in the computation are governed by the combinatorics of the Berkovich convex hull of the zeros and poles of the coordinate functions. In order to illustrate this general idea, we begin with an example involving a parametrized rational curve in \mathbf{G}_m^2 .

Example 8.7. Let $K = \mathbf{C}_p$ for some prime p (with $\text{val}(p) = 1$) and let $X \subset \mathbf{G}_m^2$ be the algebraic curve from Example 2.6 given parametrically by $x(t) = t(t - p)$, $y(t) = t - 1$ for $t \in K \setminus \{0, 1, p\}$. One way to compute $\text{Trop}(X)$ would be to use (classical) elimination theory to find an equation F for $\text{Trop}(X)$ and then use the Newton complex of F to plot the tropicalization. We will take a different approach, computing $\text{Trop}(X)$ (with multiplicities) directly and then using our knowledge of $\text{Trop}(X)$ to determine F by linear algebra. (As mentioned in the introduction, this idea is known as ‘tropical implicitization’.)

Let $\Sigma \subset \mathbf{P}_{\text{an}}^1$ be the minimal skeleton of $\mathbf{P}^1 \setminus \{0, 1, p, \infty\}$, the projective line punctured at the zeros and poles of x and y . It is very easy to determine Σ using Berkovich’s concrete description of \mathbf{A}_{an}^1 in terms of nested sequences of balls (cf. (2.4) and [Ber90, p.18]): if $\zeta_{a,r}$ is the point of \mathbf{P}_{an}^1 corresponding to the closed ball of radius r around a , then Σ is the union of the following five embedded segments/rays in \mathbf{P}_{an}^1 (see Figure 1):

$$\begin{aligned} e_1 &= \{\zeta_{p,|p|^\alpha} : \alpha \geq 1\}, e_2 = \{\zeta_{0,|p|^\alpha} : 0 \leq \alpha \leq 1\}, e_3 = \{\zeta_{0,|p|^\alpha} : \alpha \geq 1\}, \\ e_4 &= \{\zeta_{1,|p|^\alpha} : \alpha \geq 0\}, e_5 = \{\zeta_{0,|p|^\alpha} : \alpha \leq 0\}, \end{aligned}$$

as shown in Example 2.6.

Recall that trop factors through the retraction of $\mathbf{P}_{\text{an}}^1 \setminus \{0, 1, p, \infty\}$ onto Σ . It is easy to work out the tropicalization of the interior e_i° of each of the five edges of Σ , and by continuity we obtain the tropicalization of the endpoints. For example, to work out $\text{Trop}(e_1^\circ)$ (the ray from $\zeta_{p,|p|}$ to p) we need to find $(\text{val}(x(\zeta)), \text{val}(y(\zeta)))$ for $\zeta = \zeta_{p,|p|^\alpha}$ with $\alpha > 1$. This is the same as

$$\{(\text{val}(x(t)), \text{val}(y(t))) : t \in K, \text{val}(t - p) > 1\}.$$

The ultrametric inequality implies that if $\alpha = \text{val}(t - p) > 1$ then $\text{val}(x(t)) = \alpha + 1$ and $\text{val}(y(t)) = 0$. Thus

$$\text{Trop}(e_1) = \{(\alpha + 1, 0) : \alpha \geq 1\}.$$

Similarly, for the bounded edge e_2 of Σ we have

$$\text{Trop}(e_2) = \{(2\alpha, 0) : 0 \leq \alpha \leq 1\}.$$

Continuing in this way, we find that $\text{Trop}(X) \subset \mathbf{R}^2$ is a fan consisting of 3 rays emanating from the origin in the directions $(1, 0)$, $(0, 1)$, $(-2, -1)$. The expansion factor along e_2 is 2 and the other expansion factors along edges of Σ are equal to 1. Using Corollary 6.9, we see that the ray in the direction $(1, 0)$ has tropical multiplicity 2 while the other two rays have tropical multiplicity 1. (This is a good illustration of the power of Corollary 6.9, since there is no obvious way to calculate tropical multiplicities for a parametrically represented curve using the definition.)

By Theorem 6.25, the tropical multiplicity at the vertex $(0, 0)$ of $\text{Trop}(X)$ is equal to 1. An explicit calculation using the implicit equation for X from Example 2.6 confirms this: the initial degeneration at $(0, 0)$ is isomorphic to the integral k -subscheme of \mathbf{G}_m^2 defined by $y^2 + 2y = x - 1$. By Theorem 6.8 and Corollary 6.9, the tropical multiplicity at each point $w = (2\alpha, 0)$ with $0 < \alpha < 1$ is equal to the expansion factor along e_2 , which is 2. An explicit calculation shows that the initial degeneration at any point $w = (\beta, 0)$ with $\beta > 0$ is isomorphic to the length 2 non-reduced k -subscheme of \mathbf{G}_m^2 defined by $(y + 1)^2 = 0$, so in fact every point along the positive x -axis in \mathbf{R}^2 has tropical multiplicity 2.

8.8. Tropical implicitization. The method of drawing parametrized curves illustrated in the previous section gives a fairly robust approach to the problem of ‘tropical implicitization’ (see [ST08, STY07]). We illustrate the idea with a continuation of Example 2.6.

Example 8.9. As in Example 2.6, let $K = \mathbf{C}_p$ and let $X \subset \mathbf{G}_m^2$ be the algebraic curve given parametrically by $x(t) = t(t - p)$ and $y(t) = t - 1$. We wish to compute a defining polynomial $f(x, y)$ for X . (Of course, in this example one can easily do the required elimination theory by hand; however, in more complicated examples it can be very time-consuming to solve the elimination problem but the present method may still work.) The point is that since we know $\text{Trop}(X)$ as a weighted polyhedral complex, we can work out the Newton complex of f (up to translation) by [EKL06, Theorem 2.1.1]. In this case both the Newton polygon and the Newton complex of f are the triangle T with vertices $(0, 0)$, $(1, 0)$, $(0, 2)$. We therefore have $f(x, y) = y^2 + ay + bx + c$ for some $a, b, c \in K$. We can solve for the unknown coefficients a, b, c by sampling points and doing linear algebra. This yields the implicit equation $y^2 + (2 - p)y = x + (p - 1)$ for X .

Note that if one wants to recover the Newton polygon of f from $\text{Trop}(X)$, it is crucial to know the tropical multiplicities on edges of $\text{Trop}(X)$ and not just the underlying polyhedral complex (cf. (2.1) and [ST08, Remark 3.17]). Thus the formula given in Corollary 6.9 is important for applications to tropical implicitization.

8.10. Using potential theory on graphs to draw tropical curves. For a curve X of genus at least 1, one can in principle still apply the method from (8.6). However, in practice it is almost always easier to use the Slope Formula (Theorem 5.69), which is a convenient computational shortcut for working out tropicalizations of curves. Since this formula is useful even in the genus zero case, we first illustrate the idea by revisiting the parametrized rational plane curve from Example 2.6.

Example 8.11. Recall that $X \subset \mathbf{G}_m^2$ in Example 2.6 is the algebraic curve given parametrically by $x(t) = t(t - p)$, $y(t) = t - 1$ and Σ is the minimal skeleton of $\mathbf{P}^1 \setminus \{0, 1, p, \infty\}$. Let $\zeta = \zeta_{0,1}$, $\zeta' = \zeta_{0,1/p}$. By the Slope Formula, since $\text{div}(x) = (0) + (p) - 2(\infty)$, $\text{val}(x) = -\log|x|$ is the unique piecewise linear function F on Σ having slope -2 along the ray $[\zeta, \infty)$, slope 1 along the rays $[\zeta', 0)$ and $[\zeta', p)$, slope 2 along the segment $[\zeta, \zeta']$, and slope 0 along the ray $[\zeta, 1)$. Similarly, since $\text{div}(y) = (1) - (\infty)$, we see that $\text{val}(y) = -\log|y|$ has slope -1 along the ray $[\zeta, \infty)$, slope 1 along the ray $[\zeta, 1)$ and slope 0 elsewhere on Σ . It follows easily that (up to an additive translation) $\text{Trop}(X)$ is the tropical curve depicted in Figure 1. Note that we obtain the multiplicities of the edges and rays in $\text{Trop}(X)$ without any extra work by using Corollary 6.9.

Example 8.12. We now consider the tropicalization of \mathbf{P}^1 associated to the morphism $\mathbf{P}^1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ given by $x(t) = t^2(t - 1)^2(t - p^2)$ and $y(t) = t(t - 1)(t - p)$, shown in Figure 11. Let Σ be the minimal skeleton of \mathbf{P}^1 punctured at the zeros and poles of x and y . In this example, $\text{Trop}(X) = \text{Trop}(\Sigma)$ has first Betti number 1; this is due to the fact that the tropicalization map is not injective on Σ . (If the coordinate function $z(t) = t$ is added, then Σ maps homeomorphically and isometrically onto its 3-dimensional tropicalization and the ‘fake homology’ disappears; this is a concrete illustration of the intuitive idea behind Theorem 1.2.) It is a useful exercise to work out the picture depicted in Figure 11 using the Slope Formula.

Example 8.13. We explain how to draw $\text{Trop}(E)$ for the elliptic curve \widehat{E}/K from Example 2.8, defined by the Weierstrass equation $y^2 = x^3 + x^2 + t^4$ over the completion K of $\mathbf{C}\{\{t\}\}$ (see Figure 3). The j -invariant of \widehat{E} has valuation -4 , so \widehat{E} has multiplicative reduction and we know that the minimal skeleton Σ of \widehat{E} is isometric to a circle of length 4. One can see the circle of length 4 without using Tate’s nonarchimedean uniformization theory from the fact that the given Weierstrass equation defines a semistable model \mathcal{E} for \widehat{E} whose special fiber is a projective line with a node at $(0, 0)$, and a local analytic equation for the point $(0, 0) \in \mathcal{E}$ is $x''y'' = t^4$, where $x' = x\sqrt{1+x}$, $x'' = y - x'$, and $y'' = y + x'$.

The rational function x on \widehat{E} has degree 2 and its divisor is $(Q_1) + (Q_2) - 2(\infty)$, where $Q_1 = (0, t^2)$ and $Q_2 = (0, -t^2)$. The rational function y on \widehat{E} has degree 3 and its divisor is $(P_1) + (P_2) + (P_3) - 3(\infty)$, where $\text{val}(x(P_1)) = \text{val}(x(P_2)) = 2$ and $\text{val}(x(P_3)) = 0$. The points P_1, P_2 reduce to the node $(0, 0) \in \mathcal{E}(k)$ and P_3 reduces to the smooth point $(-1, 0) \in \mathcal{E}(k)$.

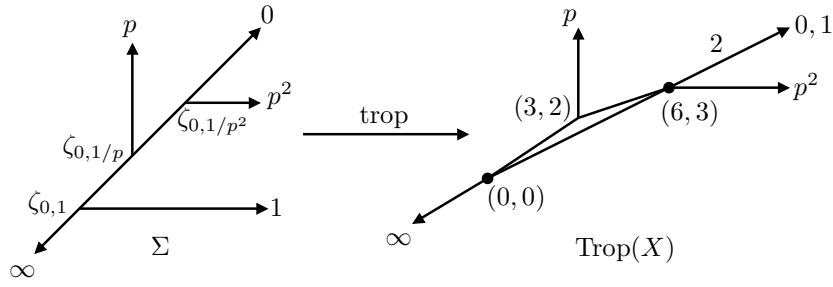


FIGURE 11. The skeleton and tropicalization of the curve $X \cong \mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^1$ given by the implicit equations $(t^2(t-1)^2(t-p^2), t(t-1)(t-p))$ from Example 8.12. Here the first Betti number of $\text{Trop}(X)$ is equal to 1 even though X^{an} is contractible.

Let E be the affine curve obtained from \widehat{E} by puncturing at the six zeros and poles of x and y , and let $\Gamma \subset E^{\text{an}}$ be its minimal skeleton. One checks by explicit computations that Γ is as depicted in Figure 3. (For this, it is helpful to know that, identifying the circle Σ with the interval $[0, 4]$ with its two endpoints glued together, the retraction $\tau_\Sigma(P)$ of a point $P = (x, y)$ in the formal fiber over the node to Σ is given explicitly by $(x, y) \mapsto \text{val}(x'') \pmod{4}$.) Let $A = \tau_\Sigma(\infty)$ and $B = \tau_\Sigma(Q_1)$ be the two branch points of Γ belonging to Σ and let e, e' be the two segments in Σ from A to B .

By the Slope Formula, the function $\text{val}(x) = -\log |x|$ has slope 1 along e and e' , slope 1 along the rays from B to Q_1 and Q_2 , slope -2 along the ray from A to ∞ , and slope 0 elsewhere on Σ . Similarly, the function $\text{val}(y) = -\log |y|$ has slope 1 along e and e' , slope 1 along the ray from A to P_3 and along the rays from B to P_1 and P_2 , slope -3 along the ray from A to ∞ , and slope 0 elsewhere. From this, it is straightforward to recover the tropicalization of E as depicted in Figure 3 (where the tropical edge multiplicities are computed using Corollary 6.9).

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